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COMPARISON OF THREE SERVO-MECHANISM DESIGN
TECHNIQUES FOR ACCOMMODATION OF AN EXTERNAL
DISTURBANCE

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This report contains a comparison of two techniques proposed by Johnson and a technique proposed by Davison for design of a servo-mechanism controller which will accommodate an external disturbance. A controller is designed, using each of the three techniques, for an example plant and external disturbance. Plots are presented to show the performance of each controller. Plots showing the sensitivity of each controller to variations in controller gains and to disturbance inputs different from the assumed disturbance input are also presented.

EXECUTIVE SUMMARY

This report illustrates the application of three different procedures for design of a servo-mechanism controller which will track input servo-commands and, at the same time, accommodate an external disturbance input. Two of the procedures were proposed by Johnson [1, 3, 6]; one uses optimal control techniques and the other uses a linear algebraic approach. The third technique, for design of a robust controller, was proposed by Davison [8, 9].

These three techniques were applied to an example consisting of a linear, time-invariant second-order plant with servo-command inputs of the form $y_c = c_0 + c_1 t$ and an external disturbance input of $w = e^t$. A servo controller was designed using each technique to demonstrate the various steps involved in each approach. The performance of the resulting controllers was then investigated.

The simplest controller design was found to be that resulting from the application of Johnson's linear algebraic approach. The controller designed using Johnson's optimal control approach was the most computationally complex of the three techniques and was also the most sensitive to variations in the system gains. The controller resulting from Davison's approach required the most additional integrators for its implementation and gave the worst transient response to an initial condition. All of the controllers were found to be sensitive to differences between the actual and modelled disturbance input to the plant, with the controller from Davison's technique being the least sensitive. All of the controllers demonstrated good servo-tracking performance in the absence of external disturbances and all were able to accommodate the effects of the external disturbance out to a run time of about 10 seconds.

CONTENTS

LIST OF FIGURES

LIST OF FIGURES - Continued

LIST OF FIGURES-Continued

LIST OF TABLES

I. INTRODUCTION

Disturbances are defined as the uncontrollable inputs which act on a dynamical system. There are many varieties of disturbance inputs which can be associated with a controlled system and they are, for the most part, completely unpredictable in magnitude and in their arrival times.

Johnson [1-7,10] introduced the idea of mathematically describing uncertain waveform-structured disturbances by representing them as a weighted linear combination of known basis functions of the form

$$w(t) = \sum_{i=1}^{n} c_i f_i(t), \tag{1}$$

where $w(t)$ is the plant disturbance vector and is a p-vector, and the weighting coefficients c_i are completely unknown constants which can change in magnitude in a random, once-in-a-while fashion. The basis functions $f_i(t)$ are completely known because they are chosen by the designer based on the waveform patterns exhibited (or thought to be exhibited) by the disturbance.

Johnson also proposed [1,3,6] two systemmatic procedures for designing multivariable servomechanism controllers which can operate in the face of these unknown, waveform-type external disturbances and unknown waveform-type servo-command inputs. Davison, et al., [8,9] proposed alternative design procedures for the same class of servomechanism control problems.

In this report, these design techniques will be applied to an example plant in order to compare the controller design procedures. The main steps in each design procedure are listed. The performance of each of the controllers is examined and the results presented. In addition, the sensitivity of each of the controllers to variations in the associated control gains and to mismatches between the actual disturbance acting on the plant and the disturbance modelled in the design process are investigated.

1

II. PLANT, SERVO-COMMAND AND EXTERNAL DISTURBANCE GENERAL MODELS

The systems considered in this report are described in two different ways to maintain consistency with the nomenclature in the references where the design techniques are presented.

For use in the two design techniques proposed by Johnson, these systems are represented as follows. The plant is described by equations of the general form

$$\dot{x}(t) = Ax(t) + Bu(t) + Fw(t) \tag{2}$$

$$y(t) = Cx(t) + Eu(t) + Gw(t) \tag{3}$$

where $x(t)$ is the plant state vector and is an n-vector, $u(t)$ is the plant control input vector and is an r-vector, $w(t)$ is the plant disturbance vector and is a p-vector, $y(t)$ is the plant output vector and is an m-vector and A, B, F, C, E and G are appropriate size, known matrices with time-invariant elements.

The general form of the disturbance state model is

$$w(t) = Hz(t) + Lx(t) \tag{4}$$

$$\dot{z}(t) = Dz(t) + Mx(t) + \sigma(t) \tag{5}$$

where $z(t)$ is the ρ-dimensional disturbance state vector, $\sigma(t)$ is a sparsely populated vector impulse sequence and H, L, D and M are appropriate size, known matrices.

The general form of the servo-command state model [3] is

$$y_c(t) = G_c c(t) \tag{6}$$

$$\dot{c}(t) = E_c c(t) + \mu(t) \tag{7}$$

where $c(t)$ is the ν-dimensional servo-command state vector, $\mu(t)$ is a sparsely populated vector impulse sequence and G_c and E_c are appropriate size, known matrices.

For use with the design technique proposed by Davison, these systems are represented as follows. The plant, in this case, must be linear and time-invariant and is described as [8]

$$\dot{x} = Ax + Bu + E\omega \tag{8}$$

$$y = Cx + Du + F\omega \tag{9}$$

where ω represents the disturbance vector and is an Ω-vector, x is an n-vector, u is an m-vector and y is an r-vector and is the output which is to be regulated. The error in the system is the difference between the output y and the specified reference input y_{ref} and is described as [9]

$$e = Cx + Du + F\omega - Gy_{ref}. \tag{10}$$

The disturbance vector is described as [8]

$$\dot{z}_1 = A_1 z_1, \tag{11}$$

$$\omega = C_1 z_1, \tag{12}$$

where (C_1, A_1) is observable, $z_1(0)$ may or may not be known, and z_1 is an n_1-vector. The specified reference input vector is described as [8]

$$y_{ref} = G\sigma \tag{13}$$

$$\dot{z}_2 = A_2 z_2 \tag{14}$$

$$\sigma = C_2 z_2 \tag{15}$$

where (C_2, A_2) is observable, $z_2(0)$ is known and z_2 is an n_2-vector.

3

III. MODELS FOR THE EXAMPLE PLANT

The example [12] to which each of the design techniques is to be applied is as follows. The plant is described by

$$\ddot{y} - y = u + w . \tag{16}$$

The external disturbance which is assumed to act on the plant is described by

$$\dot{w} - w = 0 , \tag{17}$$

almost everywhere. The servo-command input to the plant is given as

$$\ddot{y}_c = 0 , \tag{18}$$

almost everywhere.

In terms of the state-space models given in Section II, the differential equations (16), (17), and (18) can be represented as follows. For the plant, let

$$x_1 = y \tag{19}$$

$$x_2 = \dot{y} . \tag{20}$$

Then, from Equation (16), one has

$$\dot{x}_2 = \ddot{y} = y + u + w = x_1 + u + w \tag{21}$$

and it follows that, in terms of Equations (2), (3), (8), and (9), the plant model is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w \tag{22}$$

$$y = (1, 0)x . \tag{23}$$

The external disturbance given by Equation (17) represents an exponential type disturbance, i.e.,

$$w(t) = ce^t . \tag{24}$$

If

$$z = w = ce^t , \tag{25}$$

then

$$\dot{z} = \dot{w} = ce^t = z \tag{26}$$

4

and the external disturbance model can be represented in terms of Equations (4) and (5) as

$$w = z \tag{27}$$

$$\dot{z} = z + \alpha(t) \quad . \tag{28}$$

In terms of the disturbance model given by Equations (11) and (12), the disturbance is represented as

$$\omega = z_1 \tag{29}$$

$$\dot{z}_1 = z_1 \quad . \tag{30}$$

The servo-command given by Equation (18) represents

$$\ddot{y}_c(t) = c_1 \tag{31}$$

$$y_c(t) = c_0 + c_1 t \tag{32}$$

which can be expressed in terms of Equations (6) and (7) as

$$y_c = (1, 0)c \tag{33}$$

$$\begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \mu(t) \quad . \tag{34}$$

For the model given by Equations (13) through (15), the servo-command is represented as

$$\begin{pmatrix} \dot{z}_{21} \\ \dot{z}_{22} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} z_{21} \\ z_{22} \end{pmatrix} \tag{35}$$

$$\sigma = (1, 0) \begin{pmatrix} z_{21} \\ z_{22} \end{pmatrix} \tag{36}$$

$$y_{ref} = \sigma \quad . \tag{37}$$

5

IV. BACKGROUND

A detailed description of the two design techniques proposed by Johnson can be found in [1,3,6]. In both procedures, the purpose is to design control components, where the total control is allocated as

$$u(t) = u_c + u_s , \qquad (38)$$

so that the effects of the external disturbances are absorbed (or minimized) and the primary control task is achieved. The component u_s is designed to achieve the primary control task, i.e., $y(t) \to y_c(t)$ "rapidly." The component u_c is designed to accomplish the external disturbance absorption.

In the procedure described in [1], optimal control techniques are applied and the control u_s is designed to minimize a quadratic performance index J involving the tracking error, defined as

$$e(t) = y_c(t) - y(t) , \qquad (39)$$

and the control component u_s. In the procedure described in [3], linear algebraic methods are utilized to design u_s. This technique is extended in [6] where improved computational procedures are presented for calculating the necessary feedback gains.

Both of these design techniques involve the design of a composite state observer to provide estimates of the plant, servo-command and external disturbance states for use in the controllers.

The technique proposed by Davison, et al., [8,9] is for the design of a controller which is robust to external disturbance effects and to perturbations in plant parameters and system gains. This controller requires the design of a stabilizing compensator, called a complementary controller, and a new type of compensator, called a servo-compensator. The complementary controller is a model of the plant and is designed to stabilize the closed loop system consisting of the plant/servo-compensator/complementary-controller combination. The servo-compensator is a model of the disturbance/reference inputs to the system. Its purpose is to assure that the controlled system is stabilizable and will achieve robust control.

None of the above techniques require that the external disturbance be measureable. All of the techniques use a model of the plant as part of the controller. Davison's servo-compensator can be designed without necessarily knowing the exact models for the external disturbance or reference inputs. Johnson's techniques also use a model of the external disturbance input and of the reference input, but these must be either known or assumed as known. In Davison's technique, consideration is restricted to linear, time-invariant plants. Johnson's techniques do not require that the plant be time-invariant.

6

V. DESIGN TECHNIQUE 1

In this section, the design technique proposed by Johnson in [1] is applied to the plant of Section III. As stated in Equation (38), the control u is divided into two parts: (1) u_c, which is to be designed to counteract the external disturbances and (2) u_s, which is to be designed so that y(t) follows $y_c(t)$. To achieve the condition that the output follows the reference input, u_s is to be designed to minimize a performance index J of the form [1]

$$J[u;x_0,t_0,T] = \frac{1}{2} \epsilon^T(T)S\epsilon(T) + \frac{1}{2} \int_{t_0}^{T} [\epsilon^T(t)Q(t)\epsilon(t) + u_s^T Ru_s]dt , \qquad (40)$$

where S, Q, R are positive definite matrices.

The basic steps involved in the design of the control components by this technique are as follows:

     1. Obtain the state model for the expected external disturbances in the form given by Equations (4) and (5).

     2. Obtain the state model for the expected servo-commands in the form given by Equations (6) and (7).

     3. Check for satisfaction of the complete absorbability condition for the external disturbance.

     4. If total absorption of the external disturbance is possible, choose u_c to absorb the disturbance.

     5. Implement a composite state observer which will provide accurate estimates of the plant and external disturbance states. Implement u_c by using the outputs of this observer.

     6. Design u_s by first forming a composite "state" vector

$$\tilde{x} = (x \mid c)^T . \qquad (41)$$

Next, express the servo-tracking error in terms of $\tilde{x}$ as

$$\epsilon = [-C \mid G_c]\tilde{x} , \qquad (42)$$

and rewrite the performance index given in Equation (40) as

$$J[u_s;x_0,t_0,T] = \frac{1}{2} \tilde{x}^T(T)\tilde{S}\tilde{x}(T) + \frac{1}{2} \int_{t_0}^{T} [\tilde{x}^T(t)\tilde{Q}(t)\tilde{x}(t)+u_s^T(t)Ru_s(t)]dt . \qquad (43)$$

7

7. Using standard linear-quadratic regulator theory, implement the control u_s as

$$u_s^0 = [K_1(t) \mid K_2(t)]\tilde{x} . \tag{44}$$

Steps 1 and 2 were accomplished in Section III. The disturbance state model is given by Equations (27) and (28) and the servo-command state model by Equations (33) and (34). For step 3, complete absorption of the external disturbance term, ideally, will require that

$$Bu_c(t) \cong -Fw(t) \tag{45}$$

for all admissible $w(t)$ and for all $t \geq t_0$. For this to be possible, the following absorbability condition must be satisfied:

$$R[F] \subseteq R[B] , \quad t_0 \leq t \leq T , \tag{46}$$

(where $R[\cdot]$ denotes the column range space of $[\cdot]$), i.e.,

$$\text{Rank}[B \mid F] = \text{Rank}[B] . \tag{47}$$

If this condition is satisfied, the implication is that

$$F \equiv B\Gamma \tag{48}$$

for some matrix Γ. For this example,

$$\text{Rank}[B \mid F] = \text{Rank} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 1 \tag{49}$$

$$\text{Rank}[B] = \text{Rank} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 ; \tag{50}$$

therefore, complete absorption of the external disturbance is possible and $F \equiv B\Gamma$ for

$$\Gamma = 1 . \tag{51}$$

The form chosen for u_c in [1] (step 4) is

$$u_c = \overline{G}y + \overline{H}\gamma , \tag{52}$$

where γ is the output of

$$\dot{\gamma} = \overline{D}\gamma + \overline{E}y + (\overline{T}_{12} + \Sigma C)Bu_s \tag{53}$$

and is an $(n+\rho-m)$-vector. The terms shown in Equations (52) and (53) are defined in [1] as follows:

$$\overline{D} = (\overline{T}_{12} + \Sigma C)(AT_{12} - \dot{T}_{12}) + \overline{T}_{22}(DT_{22} - \dot{T}_{22}) \tag{54}$$

8

$$\overline{E} = -\overline{D}\Sigma + (\overline{T}_{12} + \Sigma C)(A(C^+)^T - (\dot{C}^+)^T) + \dot{\Sigma} \tag{55}$$

$$\overline{H} = -\Gamma H T_{22} . \tag{56}$$

$$\overline{G} = -\overline{H}\Sigma \tag{57}$$

$$\overline{T}_{12} = (T_{12}^T T_{12} + T_{22}^T T_{22})^{-1} T_{12}^T \tag{58}$$

$$\overline{T}_{22} = (T_{12}^T T_{12} + T_{22}^T T_{22})^{-1} T_{22}^T . \tag{59}$$

The matrices T_{12} and T_{22} are a pair of once-differentiable $nx(n+\rho-m)$ and $mx(n+\rho-m)$ matrices, respectively, such that

$$[C \mid 0][T_{12}^T \mid T_{22}^T]^T \equiv 0 \tag{60}$$

and

$$\text{Rank}[T_{12}^T \mid T_{22}^T] \equiv n+\rho-m . \tag{61}$$

The matrices T_{12}, T_{22}, $\overline{T}_{12}$, $\overline{T}_{22}$ are part of the structure of a reduced-order observer.

The reduced-order observer (step 5 above) which provides estimates of the plant and disturbance states for use by the controller is given in [1] as follows. Define the variable

$$\epsilon(t) = -\xi(t) + \Sigma y(t) + \overline{T}_{12} x(t) + \overline{T}_{22} z(t) \tag{62}$$

and let the plant state estimate $\hat{x}(t)$ and the disturbance state estimate $\hat{z}(t)$ be defined as

$$\hat{x}(t) = [(C^+)^T - T_{12}\Sigma]y(t) + T_{12}\xi(t) \tag{63}$$

$$\hat{z}(t) = T_{22}(\xi(t) - \Sigma y(t)) . \tag{64}$$

The parameter $\xi(t)$ is governed by the equation [1]

$$\dot{\xi} = (D + \Sigma H)\xi + [(\overline{T}_{12} + \Sigma C)(A(C^+)^T - (\dot{C}^+)^T)$$

$$- (D + \Sigma H)\Sigma + \dot{\Sigma}]y + (\overline{T}_{12} + \Sigma C)Bu$$

$$= (D + \Sigma H)\xi + \Psi y + \Omega u, \tag{65}$$

with

$$D = [\overline{T}_{12} \mid \overline{T}_{22}] \left\{ \begin{bmatrix} A & FH \\ \hline 0 & D \end{bmatrix} \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix} - \begin{bmatrix} \dot{T}_{12} \\ \dot{T}_{22} \end{bmatrix} \right\} \tag{66}$$

9

$$H = [C \mid 0] \left\{ \begin{bmatrix} A & \mid FH \\ \hline 0 & \mid D \end{bmatrix} \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix} - \begin{bmatrix} \dot{T}_{12} \\ \dot{T}_{22} \end{bmatrix} \right\} \; . \tag{67}$$

The evolution equation for $\varepsilon(t)$ in Equation (62) is given [1] as

$$\dot{\varepsilon} = [\mathcal{D} + \Sigma H] \varepsilon + \overline{T}_{22} \, \sigma(t) \; . \tag{68}$$

If Σ is chosen in Equation (55) such that $\varepsilon(t) \to 0$ "rapidly," then $\hat{x}$ and $\hat{z}$ will be accurate estimates of x and w. This matrix Σ is then used in Equations (53), (54), (55) and (57).

For this example, step 5 continues as follows. From Equation (61), one has

$$\text{Rank}[T_{12}^T \mid T_{22}^T] = 2+1-1 = 2, \tag{69}$$

and from Equation (60) one has

$$[C \mid 0] \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix} = [1 \; 0 \mid 0] \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix} \equiv 0. \tag{70}$$

If T_{12} and T_{22} are chosen as

$$T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \; ; \; T_{22} = (1, \, 0) \; , \tag{71}$$

then Equation (70) is satisfied. From Equations (58) and (59), respectively, $\overline{T}_{12}$ and $\overline{T}_{22}$ can be calculated to be

$$\overline{T}_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{72}$$

$$\overline{T}_{22} = (1, \, 0)^T \; . \tag{73}$$

Making the appropriate substitutions into Equation (67) allows H to be found as

$$H = (0, \, 1) \; . \tag{74}$$

Similarly, substituting into Equation (66) gives $\mathcal{D}$ as

$$\mathcal{D} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \; . \tag{75}$$

10

Using these results, Equation (68) can be expressed as

$$\dot{\varepsilon} = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \end{bmatrix} (0,1) \right\} \varepsilon = \begin{bmatrix} 1 & \sigma_{11} \\ 1 & \sigma_{12} \end{bmatrix} \varepsilon = R\varepsilon .$$ (76)

It is desired to have $\varepsilon(t) \to 0$ "rapidly," therefore, σ_{11} and σ_{12} should be designed to accomplish this. The characteristic polynomial of R in Equation (76) is

$$\lambda^2 - (1 + \sigma_{12})\lambda - (\sigma_{11} - \sigma_{12}) = 0.$$ (77)

For good response, it is desired that this characteristic polynomial be given by

$$\lambda^2 + 16\lambda + 64 = 0 ,$$ (78)

i.e., that the characteristic roots be placed at $\lambda_1 = \lambda_2 = -8$. If Equations (77) and (78) are compared, it can be seen that one must have

$$\Sigma = \begin{pmatrix} \sigma_{11} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} -81 \\ -17 \end{pmatrix} .$$ (79)

The expression for $\dot{\xi}(t)$, Equation (65), can next be computed. The matrices Ω and Ψ are evaluated to give

$$\Omega = (\overline{T}_{12} + \Sigma C)B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} ,$$ (80)

$$\Psi = (\overline{T}_{12} + \Sigma C)[A(C^\dagger)^T - (\dot{C}^\dagger)^T] - (D + \Sigma H) \Sigma + \dot{\Sigma} = \begin{pmatrix} -1296 \\ -207 \end{pmatrix} ,$$ (81)

where $C^\dagger$ is given by

$$C^\dagger = (CC^T)^{-1}C = (1, 0) ,$$ (82)

and $\dot{\xi}$ is finally expressed as

$$\dot{\xi} = \begin{bmatrix} 1 & -81 \\ 1 & -17 \end{bmatrix} \xi + \begin{pmatrix} -1296 \\ -207 \end{pmatrix} y + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u .$$ (83)

The plant and disturbance state estimates $\hat{x}$ and $\hat{z}$ are then given as shown in Equations (63) and (64), respectively, as

$$\hat{x} = \begin{pmatrix} 1 \\ 17 \end{pmatrix} y + \begin{pmatrix} 0 \\ \xi_2 \end{pmatrix}$$ (84)

and

11

$$\hat{z} = \xi_1 + 81y . \tag{85}$$

To complete step 5 above, the linear dynamic device represented by Equation (53) must also be implemented. From Equations (54) through (57), one has

$$\overline{D} = \begin{bmatrix} 1 & -81 \\ 0 & -17 \end{bmatrix} \tag{86}$$

$$\overline{E} = \begin{pmatrix} -1296 \\ -288 \end{pmatrix} \tag{87}$$

$$\overline{H} = (-1, 0) \tag{88}$$

$$\overline{G} = -81 , \tag{89}$$

therefore, Equation (53) can be expressed as

$$\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \end{pmatrix} = \begin{bmatrix} 1 & -81 \\ 0 & -17 \end{bmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} + \begin{pmatrix} -1296 \\ -288 \end{pmatrix} y + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_s . \tag{90}$$

The output of Equation (90) is then used, along with the plant output y, to implement u_c, as shown in Equation (52), as

$$u_c = -81y - \gamma_1 . \tag{91}$$

The next step in the design procedure is step 6 above, the design of the control u_s. In Equation (41), a composite "state" vector was defined as $\tilde{x} = (x \mid c)^T$, i.e., a composite of the plant and servo-command states. The performance index J, which u_s is designed to minimize was given by Equation (43) in terms of this composite state vector. In Equation (43), the matrices $\tilde{S}$ and $\tilde{Q}$ are defined as

$$\tilde{S} = [-C \mid G_c]^T S[-C \mid G_c] \tag{92}$$

$$\tilde{Q} = [-C \mid G_c]^T Q[-C \mid G_c] . \tag{93}$$

For the example in this report, S and Q will be chosen as

$$S = Q = I . \tag{94}$$

The gain matrices K_1 and K_2 shown in Equation (44) are given by [1,10]

$$K_1 = -R^{-1} B^T K_x x \tag{95}$$

$$K_2 = -R^{-1} B^T K_{xc} c \tag{96}$$

12

and the matrices K_c, K_{xc} are solutions of the coupled matrix differential equations

$$\dot{K}_x = (-A + BR^{-1} B^T K_x)^T K_x - K_x A - C^T \tilde{Q} C; \quad K_x(T) = C^T \tilde{S} C \tag{97}$$

$$\dot{K}_{xc} = (-A + BR^{-1} B^T K_{xc})^T K_{xc} - K_{xc} E + C^T \tilde{Q} G_c; \quad K_{xc}(T) = -C^T \tilde{S} G_c. \tag{98}$$

Equations (97) and (98) can be solved once and for all by integrating in backward time, starting at $t = T$, using the initial conditions indicated. Expanding Equation (97) results in

$$\begin{bmatrix} \dot{k}_{x1} & \dot{k}_{x2} \\ \dot{k}_{x3} & \dot{k}_{x4} \end{bmatrix} = (1/r) \begin{bmatrix} k_{x3}(k_{x3}-r)-r-rk_{x2} & k_{x4}(k_{x3}-r)-k_{x1}r \\ k_{x4}k_{x3}-rk_{x1}-rk_{x4} & k_{x4}^2-rk_{x2}-rk_{x3} \end{bmatrix} \tag{99}$$

and expanding Equation (98) results in

$$\begin{bmatrix} \dot{k}_{xc1} & \dot{k}_{xc2} \\ \dot{k}_{xc3} & \dot{k}_{xc4} \end{bmatrix} = (1/r) \begin{bmatrix} k_{xc3}(k_{x3}-r)+r & k_{xc4}(k_{x3}-r)-rk_{xc1} \\ k_{x4}k_{xc3}-rk_{xc1} & k_{x4}k_{xc4}-rk_{xc2}-rk_{xc3} \end{bmatrix} \cdot \tag{100}$$

For backward time integration, let

$$t = T - \tau; \quad dt = -d\tau. \tag{101}$$

Substituting Equations (101) into Equations (99) and (100) yields

$$\begin{bmatrix} \dfrac{dk_{x1}(\tau)}{d\tau} & \dfrac{dk_{x2}(\tau)}{d\tau} \\[2ex] \dfrac{dk_{x3}(\tau)}{d\tau} & \dfrac{dk_{x4}(\tau)}{d\tau} \end{bmatrix} = (-1/r) \begin{bmatrix} k_{x3}(k_{x3}-r)-r-rk_{x2} & k_{x4}(k_{x3}-r)-k_{x1}r \\ k_{x4}k_{x3}-rk_{x1}-rk_{x4} & k_{x4}^2-rk_{x2}-rk_{x3} \end{bmatrix} \tag{102}$$

and

$$\begin{bmatrix} \dfrac{dk_{xc1}(\tau)}{d\tau} & \dfrac{dk_{xc2}(\tau)}{d\tau} \\[2ex] \dfrac{dk_{xc3}(\tau)}{d\tau} & \dfrac{dk_{xc4}(\tau)}{d\tau} \end{bmatrix} = (-1/r) \begin{bmatrix} k_{xc3}(k_{x3}-r)+r & k_{xc4}(k_{x3}-r)-rk_{xc1} \\ k_{x4}k_{xc3}-rk_{xc1} & k_{x4}k_{xc4}-rk_{xc2}-rk_{xc3} \end{bmatrix}, \tag{103}$$

with the initial conditions given by

$$K_x(T) = C^T \tilde{S} C = C^T C = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \tag{104}$$

13

$$K_{xc}(T) = -C^T \tilde{S} G_c = -C^T G_c = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} . \tag{105}$$

Equations (102) and (103) were programmed in a digital simulation and runs were made, for several different values of r, to obtain steady-state gains. The results are presented in Table 1.

TABLE 1. Steady-State Optimal Gains

R	k_{x1}	$k_{x2}=k_{x3}$	k_{x4}	k_{xc1}	k_{xc2}	k_{xc3}	k_{xc4}
10.	21.231	20.488	20.243	-1.93	-2.7715	-0.95345	-1.8401
1.	3.1075	2.4192	2.1974	-1.5538	-1.7071	-0.70711	-1.0987
0.1	0.9745	0.43166	0.29382	-0.88591	-0.4833	-0.30151	-0.26711
0.01	0.47245	0.1105	0.04701	-0.46777	-0.11931	-0.0995	-0.04654
0.001	0.2556	0.032639	0.008079	-0.25537	-0.0336	-0.031607	-0.008071

To implement u_s, it is necessary to provide estimates of the servo-command states. This is done by constructing an observer for the servo-command process. As given in [1], the observer is constructed as

$$\dot{\hat{c}} = (E+NG)\hat{c} - Ny_c . \tag{106}$$

The error in the servo-command estimate is defined as

$$e_c = c - \hat{c} , \tag{107}$$

and the evolution equation for this can be found as

$$\dot{e}_c = \dot{c} - \dot{\hat{c}} = (E+NG)e_c + \mu(t) . \tag{108}$$

The gain represented by the matrix N is chosen to make $e_c \rightarrow 0$ "rapidly." For this example, one has

$$(E+NG) = \begin{bmatrix} n_1 & 1 \\ n_2 & 0 \end{bmatrix} , \tag{109}$$

and the characteristic polynomial for (E+NG) is calculated as

$$\lambda^2 - n_1 \lambda - n_2 = 0 . \tag{110}$$

14

It is desired that the roots of Equation (110) be placed at $\lambda_1 = \lambda_2 = -8$, i.e., that the characteristic polynomial be

$$\lambda^2 + 16\lambda + 64 = 0 \ . \tag{111}$$

A comparison of Equations (110) and (111), results in

$$n_1 = -16, \ n_2 = -64 \ . \tag{112}$$

Therefore, Equation (106) can be represented as

$$\begin{pmatrix} \dot{\hat{c}}_1 \\ \dot{\hat{c}}_2 \end{pmatrix} = \begin{bmatrix} -16 & 1 \\ -64 & 0 \end{bmatrix} \begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \end{pmatrix} - \begin{pmatrix} -16 \\ -64 \end{pmatrix} y_c \ , \tag{113}$$

and implemented to provide estimates of the servo-command states. The control u_s as given by Equation (44), with K_1 and K_2 from Equations (95) and (96), respectively, can now be implemented as

$$u_s^o = -R^{-1} B^T K_x \hat{x} - R^{-1} B^T K_{xc} \hat{c}$$

$$= -(1/r)(k_{x3}\hat{x}_1 + k_{x4}\hat{x}_2 + k_{xc3} \hat{c}_1 + k_{xc4} \hat{c}_2) \ . \tag{114}$$

Figure 1 is a block diagram of the overall system, including the servo-command observer, the plant, the composite plant/disturbance observer, and the control components.

This system was simulated on a digital computer and several runs were made. Table 2 is a listing of the simulation. Figure 2 shows the response of the system to an initial condition of $y(0) = 1$, for each of the values of R listed in Table 1. Figures 3 and 4 show the output $y(t)$, and the tracking error, respectively, in response to a servo-command input of $y_c = 1 + 0.1t$, with $w = 0$, for each value of R. The best performance is seen to be for R = 0.001. Figure 5 shows the system performance in response to the initial con- dition and the external disturbance $w = e^t$, with $y_c = 0$. Again, the best per- formance is with R = 0.001.

Figures 6 and 7 show the system output and tracking error, respec- tively, for a case with R=0.001, $w = e^t$ and $y_c = 1 + 0.1t$, and Figure 8 gives a plot of w versus time. Figures 9, 10, and 11 are system outputs, with R = 0.001, to servo-command inputs of 2t, -2-4t and -10+10t, respectively. As shown, the servo-tracking error remains small.

To check the sensitivity of the controller performance to variations in the gains, a set of runs were made with 5% and 10% gain variations. Figure 12 shows the servo-tracking error for a case with R = 0.001, with $y_c = 0$, $w = 0$, and with system gain variations of +5% and +10%. Figure 13 shows similar data for a case with $y_c = 1 + 0.1t$. As shown, the system response is very sen- sitive to small gain variations.

15
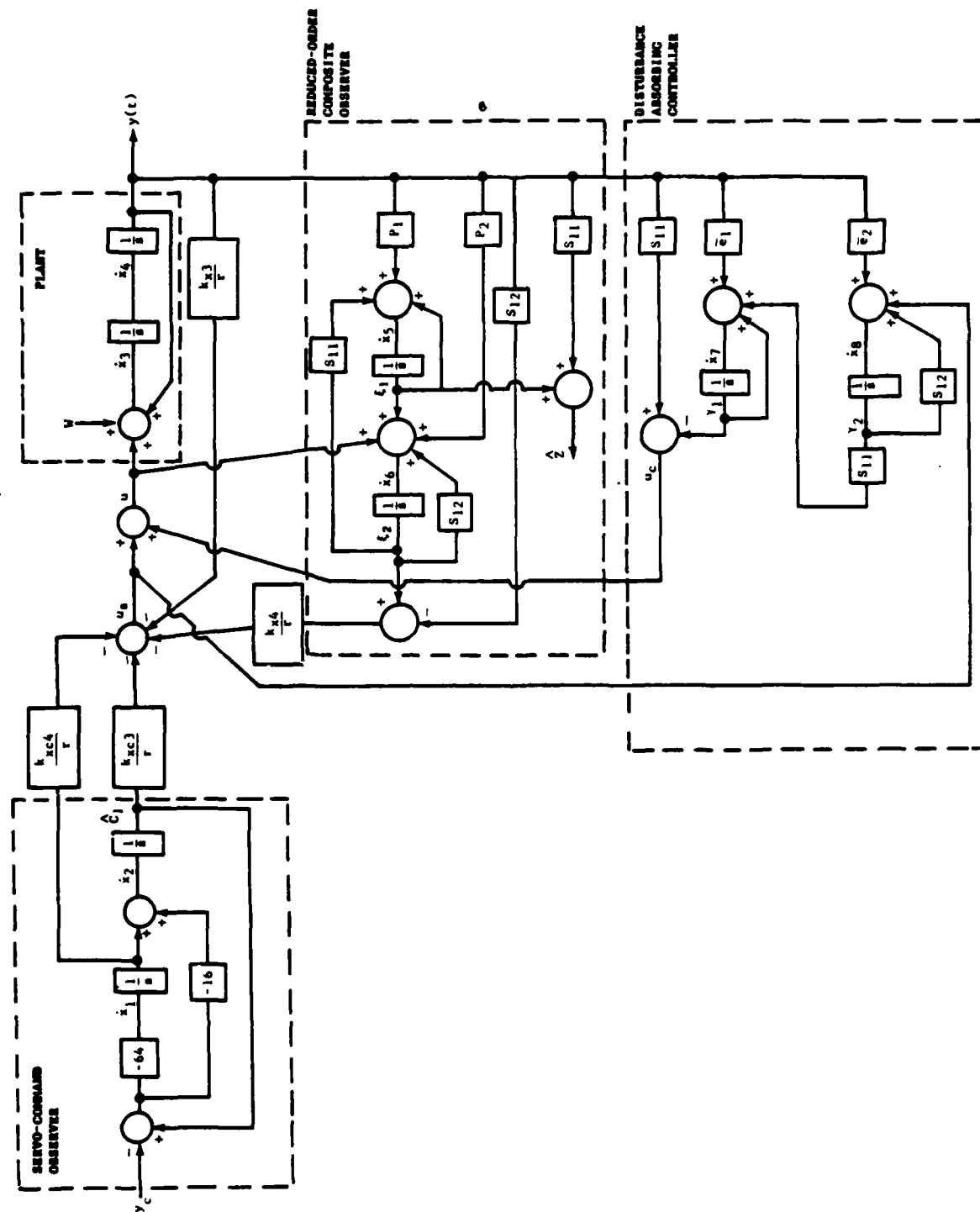
Figure 1. Composite system block diagram, technique 1.

16

TABLE 2. Simulation Listing for System of Figure 1

```
      COMMON X(8),DX(8),KUTTA,DT,NX
      DIMENSION XDAT(20)
      NVAR=16
      WRITE(20)NVAR
*********************************************************
      THIS PROGRAM SIMULATES THE EXAMPLE PLANT           *
      WITH THE CONTROL DESIGNED USING                    *
      JOHNSON'S METHOD 1                                 *
*********************************************************
      DO 100 I=1,8
      DX(I)=0.
      X(I)=0.
100   CONTINUE
      X(4) = 1.
      TIME=0.
      NX=8
      DT=0.01
      PRINT*,' ENTER R,WC0,WC1,YC0,YC1 '
      ACCEPT*,R,WC0,WC1,YC0,YC1
      PRINT*,' ENTER XKX3,XKX4,XKXC3,XKXC4 '
      ACCEPT*,XKX3,XKX4,XKXC3,XKXC4
      S11 = -81.


      S12 = -17.
      XP1 = -1296.
      XP2 = -207.
      EBAR1 = -1296.
      EBAR2 = -288.
      XR = 1./R
      UC = 0.
      US = 0.
      U = 0.
      W = 0.
      YC = 0.
      IPRT=0
1000  CONTINUE
      IF(TIME.GE.10.) GO TO 9999
      IPRT=IPRT+1
      DO 200 KUTTA=1,4
      W = WC0*EXP(WC1*TIME)
      YC = YC0 + YC1*TIME
      SCERR = X(2) - YC
      SCTK3 = -SCERR
      DX(1) = -64.*SCERR
      DX(2) = X(1) - 16.*SCERR


      US = -XR*(XKXC4*X(1)+XKXC3*X(2)+XKX3*X(4)+XKX4*(-512*X(4)+X(6)))
      UC = S11*X(4) - X(7)
      U = US + UC
      DX(3) = U + W + X(4)
      DX(4) = X(3)
      DX(5) = XP1*X(4) + S11*X(6) + X(5)
      DX(6) = X(5) + U + XP2*X(4) + 512*X(6)
      X2HAT = -512*X(4) + X(6)
      DX(7) = EBAR1*X(4) + X(7) + S11*X(8)
      DX(8) = EBAR2*X(4) + US + S12*X(8)
      GO TO (30,60,30,40),KUTTA
30    CONTINUE
      TIME=TIME+.5*DT
40    CONTINUE
60    CALL RUNK
```

TABLE 2. Simulation Listing for System of Figure 1 - Continued

```
200    CONTINUE
       TRKERR = YC - X(4)
       EXER = W - X(7)
       XDAT(1)=X(1)
       XDAT(2)=X(2)
       XDAT(3)=X(3)
       XDAT(4)=X(4)
       XDAT(5)=X(5)
       XDAT(6)=X(6)
       XDAT(7)=X(7)
       XDAT(8)=X(8)
       XDAT(9)=YC
       XDAT(10)=TIME
       XDAT(11)=US
       XDAT(12)=UC
       XDAT(13)=X2HAT
       XDAT(14)=TRKERR
       XDAT(15)=SCTKE
       XDAT(16)=W
       IF(IPRT.NE.10) GO TO 500
       WRITE(20) (XDAT(I),I=1,NVAR)
       IPRT=0
       PRINT*,' TIME = ',TIME,' Y(T) = ',X(4),' YC = ',YC
500    GO TO 1000
99     CONTINUE
       STOP
       END
       SUBROUTINE RUNK
       COMMON X(8),DX(8),KUTTA,DT,NX


       DIMENSION XA(8),DXA(8)
       GO TO (10,30,50,70),KUTTA
10     DO 20 I=1,NX
       XA(I)=X(I)
       DXA(I)=DT*DX(I)
20     X(I)=X(I)+.5*DXA(I)
       RETURN
30     TDT=2.*DT
       HDT=.5*DT
       DO 40 I=1,NX
       DXA(I)=DXA(I)+TDT*DX(I)
40     X(I)=XA(I)+HDT*DX(I)
       RETURN
50     DO 60 I=1,NX
       VDT=DT*DX(I)
       DXA(I)=DXA(I)+2.*VDT
60     X(I)=XA(I)+VDT
       RETURN
70     DO 80 I=1,NX
80     X(I)=XA(I)+(DXA(I)+DT*DX(I))/6.
       RETURN
       END
```

18
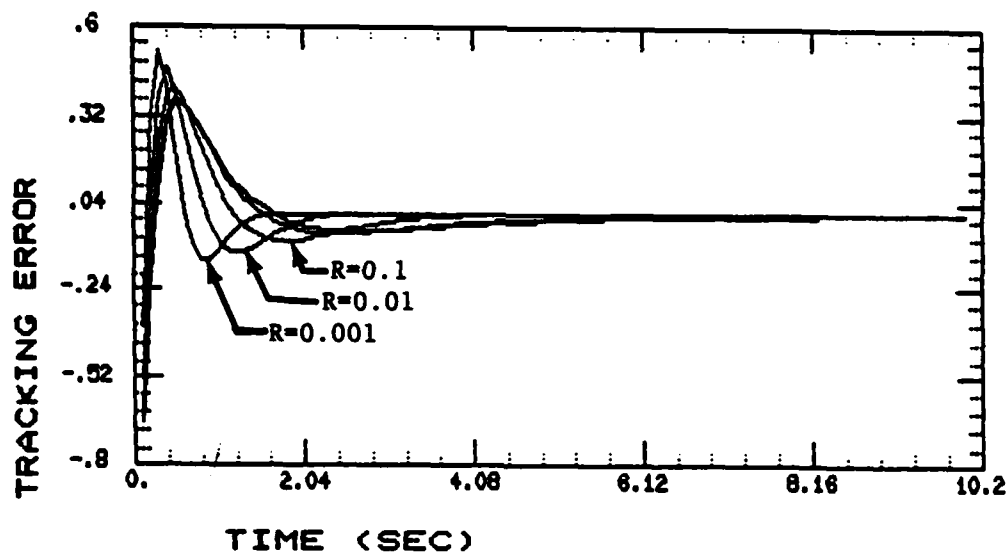
Figure 2.   Servo-tracking error, as a function of R, for a case with $y_c = 0$, $w = 0$, $y(0) = 1$.
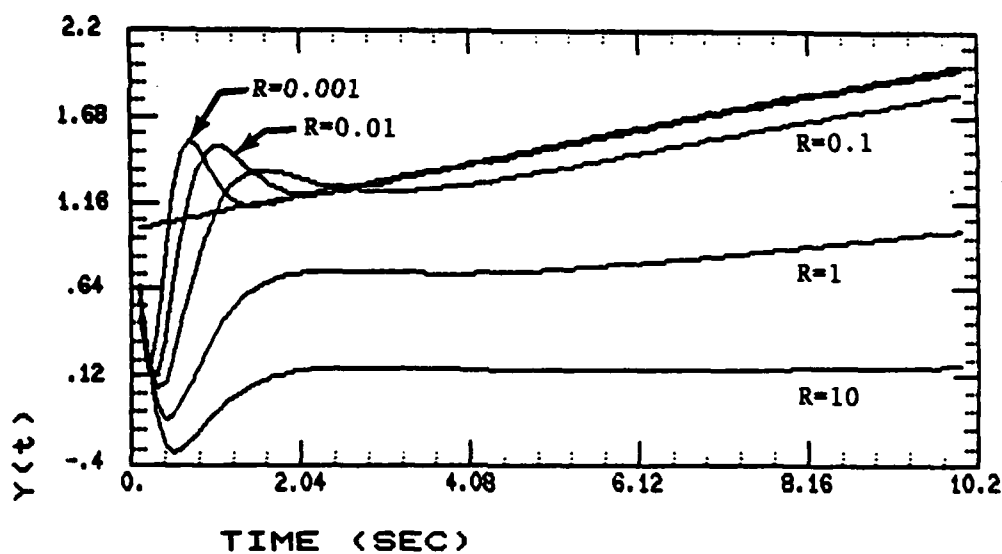
Figure 3.   Plant output response, as a function of R, for a case with $y_c = 1 + 0.1t$, $w = 0$, $y(0) = 1$.

19
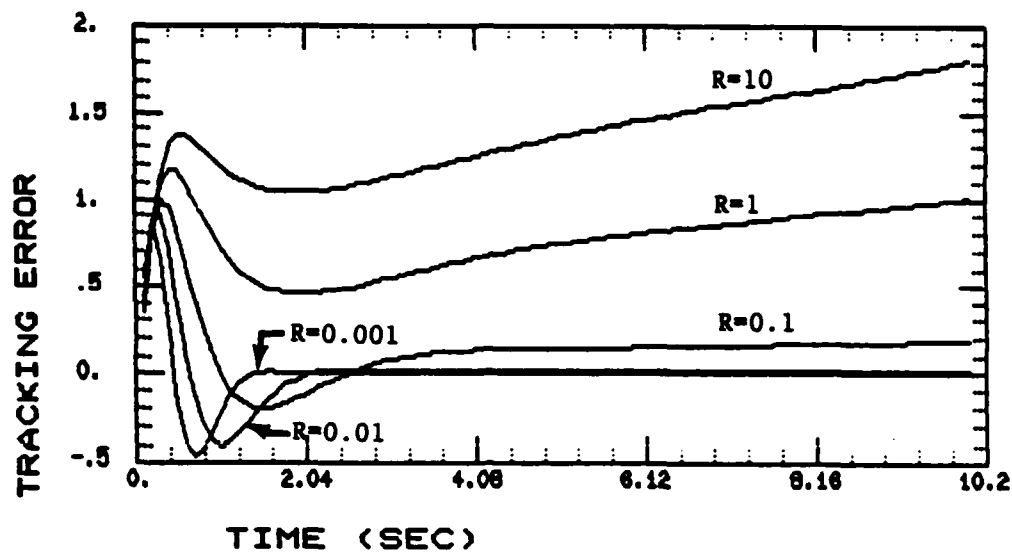
Figure 4. Servo-tracking error, as a function of R, corresponding to Figure 3.
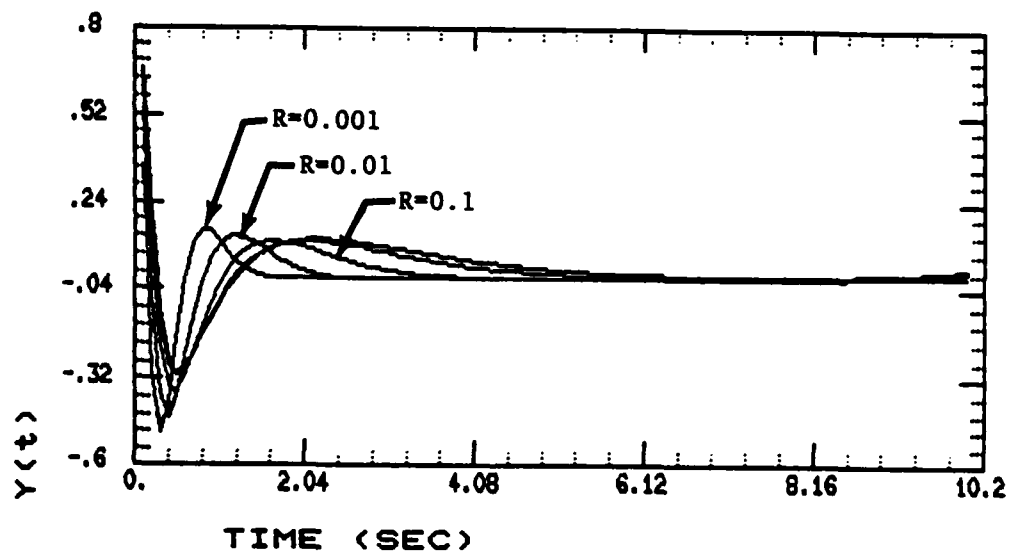
Figure 5. Plant output response, as a function of R, for a case with $y_c = 0$, $w = e^t$, $y(0) = 1$.
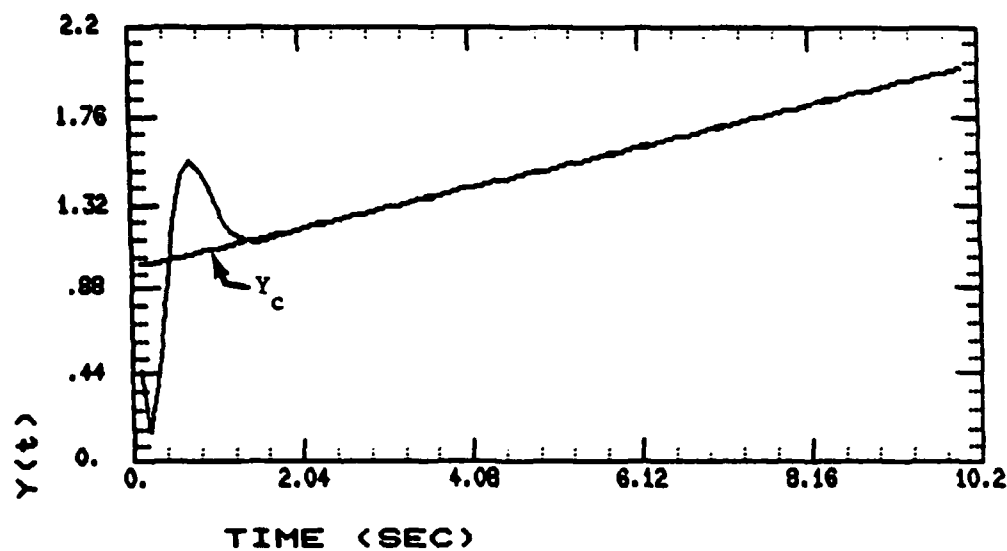
Figure 6. Plant output response, for R = 0.001, for $y_c = 1 + 0.1t$, $w = e^t$, $y(0) = 1$.
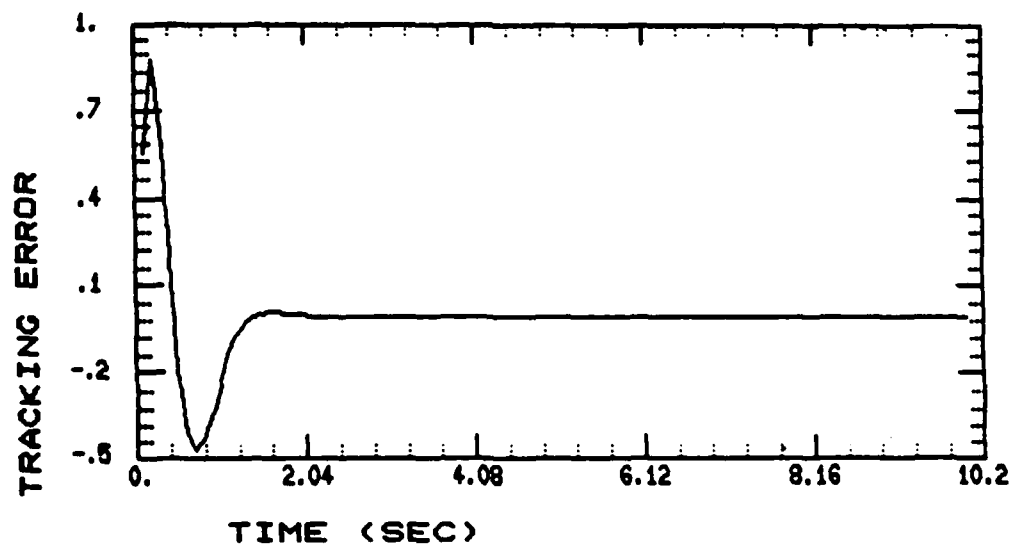
Figure 7. Servo-tracking error corresponding to Figure 6.

21
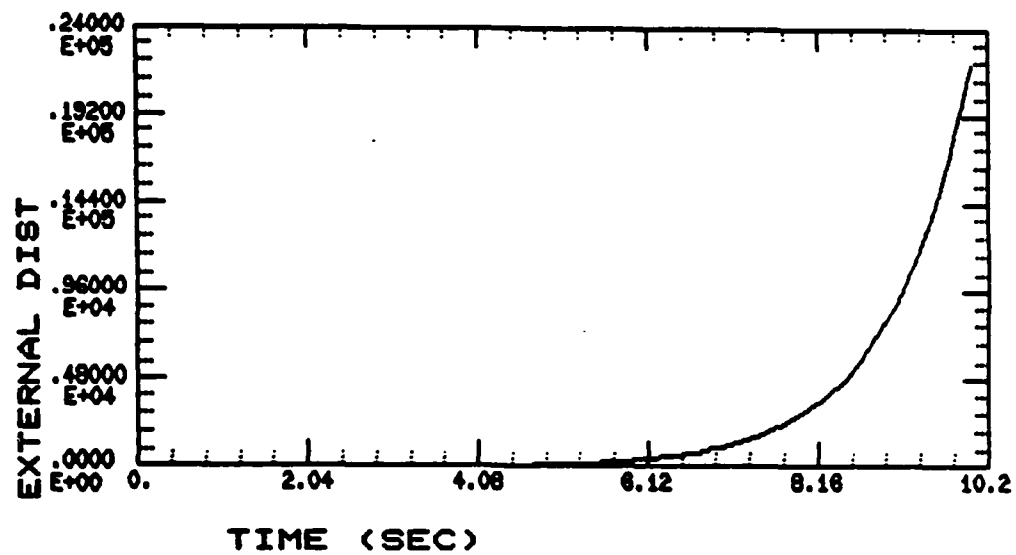
Figure 8. External disturbance magnitude.
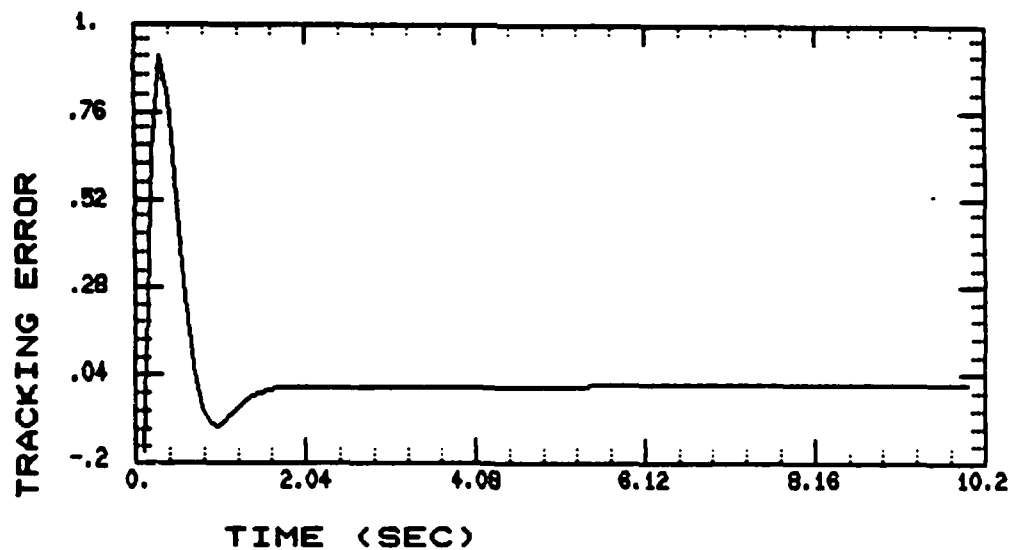
Figure 9. Servo-tracking error, with R = 0.001, for a case
with y_c = 2t, w = e^t, y(0) = 1.

22
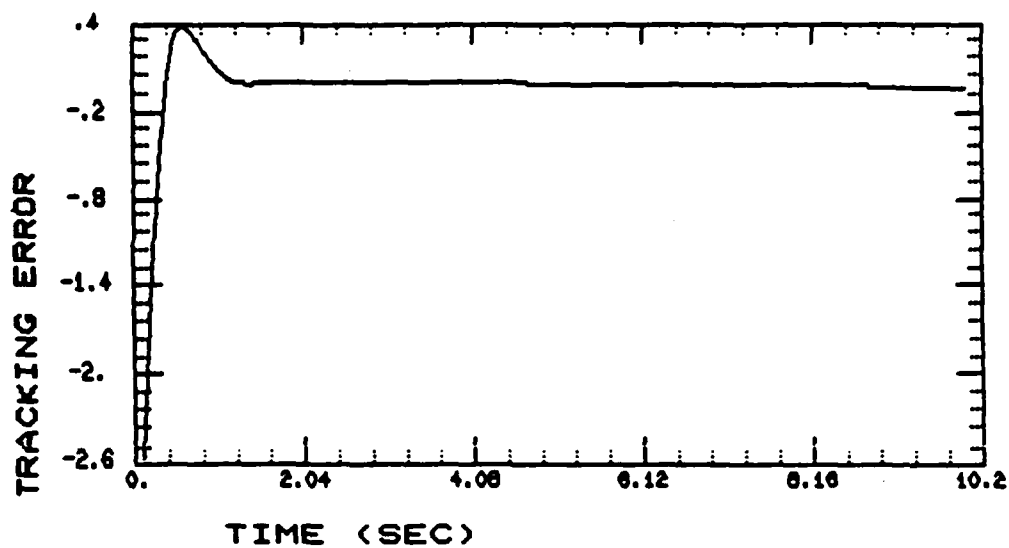
Figure 10.   Servo-tracking error, with R = 0.001, for a case
with y_c = -2 -4t, w = e^t, y(0) = 1.
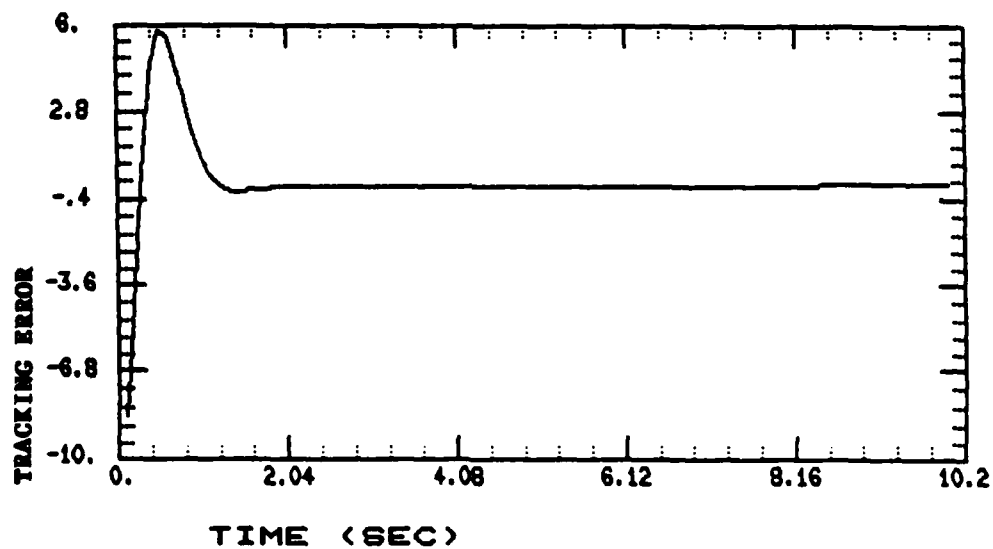
Figure 11.   Servo-tracking error, with R = 0.001, for a case with
y_c = -10 + 10t, w = e^t, y(0) = 1.

23
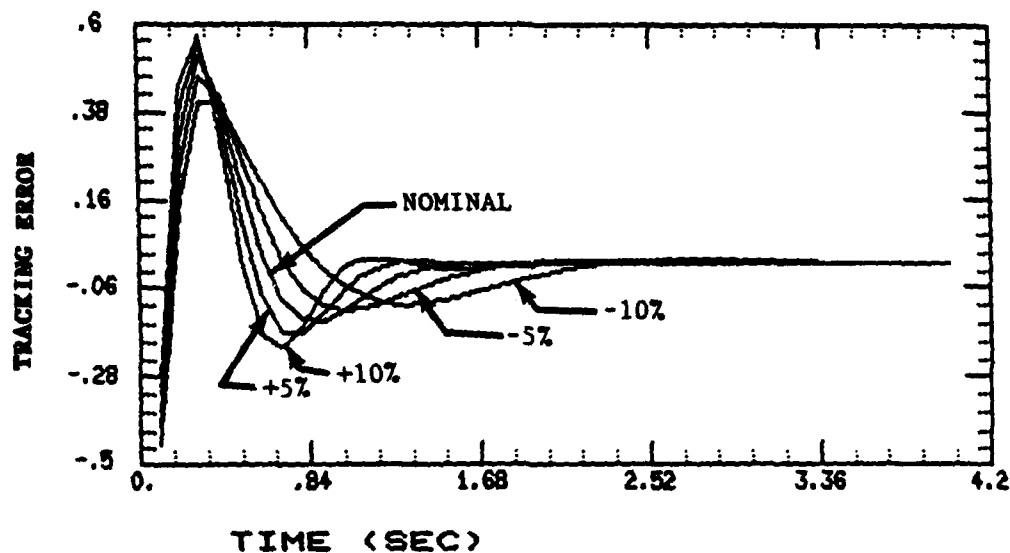
Figure 12. Servo-tracking error, with R = 0.001, with system gain variations, for a case with $y_c = 0$, $w = 0$, $y(0) = 1$.
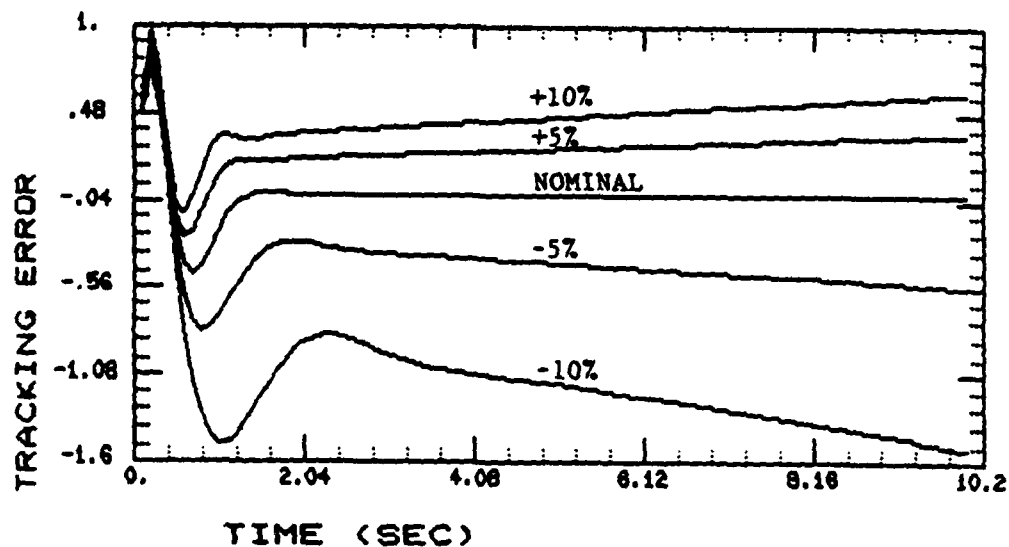
Figure 13. Servo-tracking error, with R = 0.001, with system gain variations, for a case with $y_c = 1 + 0.1t$, $w = e^t$, $y(0) = 1$.

24

To check the robustness of the controller to differences between the actual external disturbance input to the plant and that modeled in the design process, two additional runs were made. For one run, the external disturbance was $w = e^{.5t}$. For the other run, it was $w = e^{1.5t}$. Figures 14 and 15 present the results, along with a reference curve for the case with $w = e^t$. As shown, this controller is also sensitive to mismatches between the actual and assumed disturbances.
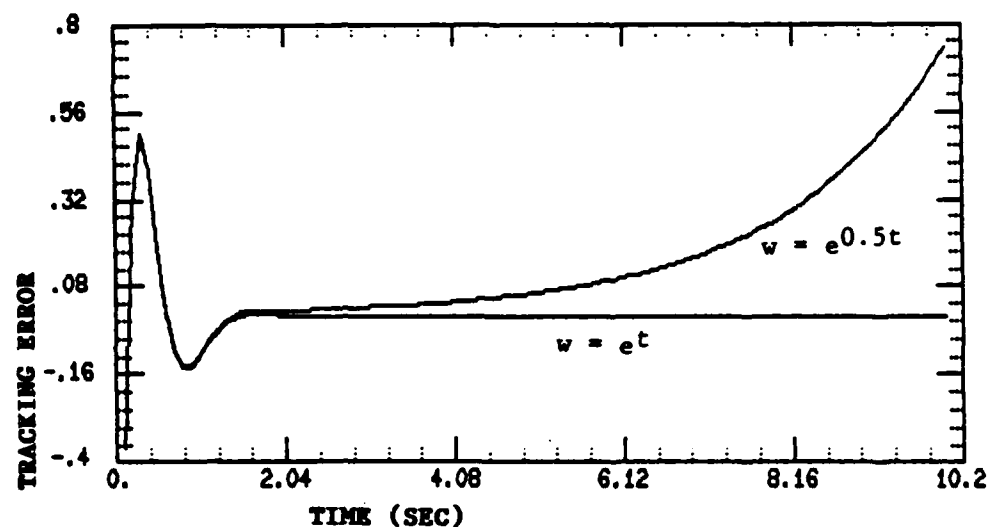
Figure 14. Servo-tracking error with $y_c = 0$, $y(0) = 1$, for cases with (1) $w = e^t$; (2) $w = e^{0.5t}$.
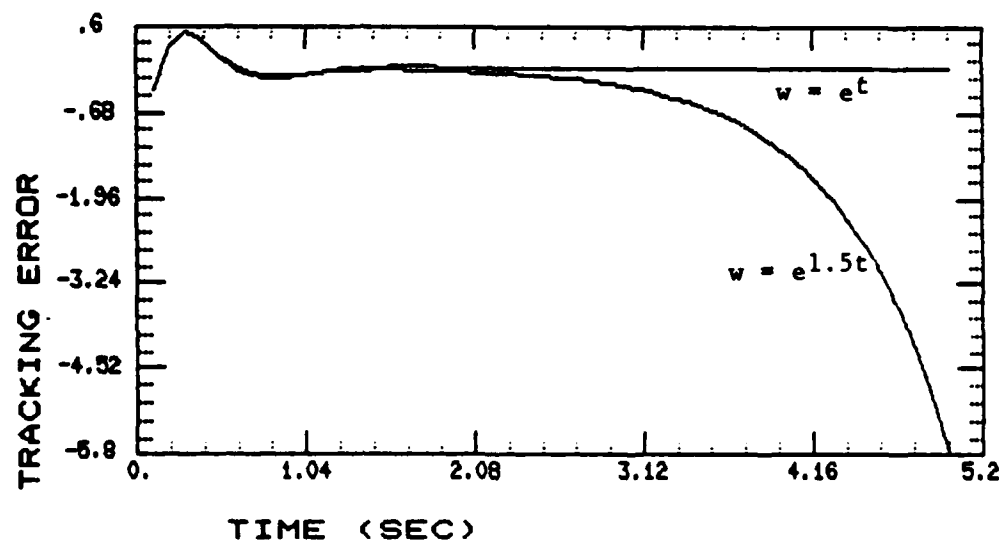
Figure 15. Servo-tracking error, with $y_c = 0$, $y(0) = 1$, for cases with: (1) $w = e^t$; (2) $w = e^{0.5t}$.

VI. DESIGN TECHNIQUE 2

In this section, the design technique proposed by Johnson in [3,6] will be applied to the plant of Section III. As was the case in the procedure of Section V, the control u is divided into two parts, u_c and u_s. These two control components have the same tasks for the design of this section as they had in the previous technique; however, the form chosen for u_c is different and u_s will be designed by using linear algebraic techniques.

The basic steps involved in this technique are as follows:

1. Obtain the state model for the expected external disturbances in the form given by Equations (4) and (5).

2. Obtain the state model for the expected servo-commands in the form given by Equations (6) and (7).

3. Check to see if the effect of the disturbances on the output can be completely absorbed, either by choosing

$$u_c = \Lambda \hat{z} \tag{115}$$

such that

$$B\Lambda + FH \equiv 0 , \tag{116}$$

or

$$C(t)\Phi(t,\tau)[B(\tau)\Lambda(\tau) + F(\tau)H(\tau)] \equiv 0 , \tag{117}$$

i.e., $(B\Lambda + FH)$ is in the null space of $C(t)\Phi(t,\tau)$, in which case the disturbance effects will be unobservable in the output space.

4. To design u_s, first check for satisfaction of the "exact trackability" condition [3] for exact servo-tracking, i.e.,

$$R[G_c(t)] \subseteq R[C(t)] , \quad t_0 \leq t \leq T . \tag{118}$$

If Equation (118) is satisfied, then it is possible to express $G_c(t)$ as

$$G_c(t) = C(t)\Theta(t) , \quad t_0 \leq t \leq T \tag{119}$$

for some (possibly non-unique) matrix $\Theta(t)$. If this condition is satisfied, then from the expression for servo error ε_y, defined as

$$\varepsilon_y = y_c - y = G_c c - Cx = G_c(\Theta c - x) , \tag{120}$$

a new variable e_{ss} is defined as [3]

$$e_{ss} = \Theta c - x . \tag{121}$$

5. The control u_s is of the form [3]

$$u_s = S_1(t)x + S_2(t)c . \tag{122}$$

Using this definition for u_s, the evolution equation for e_{ss} can be derived as [3]

$$\dot{e}_{ss} = (A+BS_1)e_{ss} + [\mathbb{E} - A\Theta + \dot{\Theta} - B(S_1\Theta + S_2)]c - B\hat{z} + \Theta\mu(t)$$

$$= (A + BS_1)e_{ss} - Vc - B\hat{z} + \Theta\mu(t) , \qquad (123)$$

where V is defined as

$$V = -\mathbb{E} + A\Theta - \dot{\Theta} + B(S_1\Theta + S_2) . \qquad (124)$$

Then, u_s (i.e., S_1, S_2) is designed so that e_{ss} rapidly approaches the null space of C. This design procedure is as follows:

    a. S_1 is designed so that for the homogeneous differential equation associated with the $\dot{e}_{ss}$ expression, $e_{ss} \to N(C)$.

    b. S_2 is designed so that the remaining terms in the $\dot{e}_{ss}$ expression which can affect e_{ss} are either zeroed out or are confined to the null space of $C\Phi$. The terms in question are given by Vc, as defined in Equations (110) and (111), since $B\hat{z}$ is in the null space of $C\Phi$ and will not affect the output. Thus, S_2 is designed so that

$$C(t)\Phi(t,\tau)[\Theta(\tau)E(\tau)-A(\tau)\Theta(\tau)+\dot{\Theta}(\tau) - B(\tau)S_1(\tau)\Theta(\tau) - B(\tau)S_2(\tau)]$$

$$= C(t)\Phi(t,\tau)V(\tau) \equiv 0 . \qquad (125)$$

    6. Finally, the state reconstructors are designed to give estimates of the plant, disturbance and servo-command states to permit practical implementation of u_c and u_s.

    Steps 1 and 2 above, were accomplished in Section III, with the disturbance state model given by Equations (27) and (28) and the servo-command model by Equations (33) and (34). For step 3, since the plant in this example is time-invariant, the requirement given by Equation (117) reduces to the requirement [3] that

$$C[\hat{B} \mid A\hat{B}] = 0, \qquad (126)$$

where

$$\hat{B} = B\Lambda + FH . \qquad (127)$$

Making appropriate substitutions into Equation (127) results in

$$\hat{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Lambda + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (128)$$

27

and it follows that

$$\Lambda = -1 \qquad (129)$$

will satisfy both Equation (116), and Equation (126). The control u_c, therefore, is chosen as

$$u_c = \Lambda z = -z \ . \qquad (130)$$

In step 4, a check must be made for satisfaction of the exact trackability condition. For this example,

$$\text{rank}[G_c \mid C] = \text{rank}[1 \ 0 \mid 1 \ 0] = 1$$
$$\qquad (131)$$
$$\text{rank}[G_c] = 1,$$

hence, Equation (118) is satisfied, and from Equation (119) it is found that

$$\theta = 1 \ . \qquad (132)$$

To proceed with step 5, the design of the control u_s, from step 5.a. above, the gain matrix S_1 is to be designed so that the solution of the homogeneous differential equation

$$\dot{e}_{ss} = (A + BS_1)e_{ss} \qquad (133)$$

approaches $N(C)$ rapidly. For this example, Equation (133) can be expanded as

$$\dot{e}_{ss} = \begin{bmatrix} 0 & 1 \\ 1+s_{11} & s_{12} \end{bmatrix} e_{ss} = A_1 e_{ss} \ , \qquad (134)$$

and the characteristic polynomial of A_1 is found to be

$$\lambda^2 - s_{12}\lambda - (s_{11} + 1) = 0 \ . \qquad (135)$$

For good response, it is desired that the characteristic roots of A_1 be located at

$$\lambda_1 = \lambda_2 = -10 \ , \qquad (136)$$

i.e., that the characteristic polynomial of A_1 be given by

$$\lambda^2 + 20\lambda + 100 = 0 \ . \qquad (137)$$

Equations (137) and (135) will be equal if

$$s_{11} = -101, \ s_{12} = -20 \ . \qquad (138)$$

28

The next step in the design procedure is step 5.b., the design of S_2. As stated earlier, S_2 must be designed so that Equation (125) is satisfied. From [3], in order to satisfy Equation (125) it is necessary and sufficient that an S_2 exist so that

$$BS_2 \equiv V + \mathcal{E} - A\theta + \dot{\theta} - BS_1\theta, \quad t_0 \leq t \leq T \tag{139}$$

for some matrix V which satisfies $C(t)\Phi(t,\tau)V(\tau) \equiv 0$, $t_0 \leq \tau \leq t$, $t_0 \leq t \leq T$. Therefore, find a V and then solve for S_2 from Equation (139). To satisfy Equation (139), V must satisfy the condition [3]

$$\text{rank}[B \mid V + \mathcal{E} - A\theta + \dot{\theta}] = \text{rank}[B] . \tag{140}$$

If Equation (140) is satisfied, then [3]

$$V + \mathcal{E} - A\theta + \dot{\theta} = B\Sigma \tag{141}$$

for some, possibly non-unique, matrix Σ.

For this example, Equation (140) is expanded as

$$\text{rank}[B \mid V + \mathcal{E} - A\theta + \dot{\theta}] = \text{rank}\left[\begin{array}{c|c} 0 & \\ 1 & \end{array} V + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\right] \tag{142}$$

$$\text{rank}[B] = \text{rank}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 , \tag{143}$$

and it can be seen that

$$\text{rank}\left[\begin{array}{c|c} 0 & \\ 1 & \end{array} V + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\right] = 1 \tag{144}$$

if V is chosen to be 0. This choice for V will also satisfy Equation (125). Given this choice for V, Equation (141) can be expanded as

$$\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \sigma_{11} & \sigma_{12} \end{bmatrix} \tag{145}$$

and therefore Σ is given by

$$\Sigma = (-1, 0) . \tag{146}$$

Next, find an S_2 which will satisfy Equation (139) by setting [3]

$$S_2 = \Sigma - S_1\theta . \tag{147}$$

From Equation (147) then, S_2 is found as

$$S_2 = (-1, 0) - (-101, -20) = (100, 20) . \tag{148}$$

The control u_s, as given by Equation (122), can now be expressed as

$$u_s = S_1 x + S_2 c = (-101, -20)x + (100, 20)c \ , \tag{149}$$

and the total control vector u is

$$u = u_c + u_s = -z - 101x_1 - 20x_2 + 100c_1 + 20c_2 \ . \tag{150}$$

For practical implementation, estimates of the plant, servo-command, and disturbance states must be used and Equation (150) implemented as

$$u = -\hat{z} - 101x_1 - 20x_2 + 100\hat{c}_1 + 20\hat{c}_2 \ . \tag{151}$$

The estimates of the plant and disturbance states will be provided by a full-order composite state observer designed using a "recipe" provided in [3]. The observer is defined by

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{z}} \end{pmatrix} = \left[\begin{array}{c|c} A + K_1 C & FH \\ \hline K_2 C & D \end{array} \right] \begin{pmatrix} \hat{x} \\ \hat{z} \end{pmatrix} - \left[\begin{array}{c} K_1 \\ \hline K_2 \end{array} \right] y + \left[\begin{array}{c} B \\ \hline 0 \end{array} \right] u \ . \tag{152}$$

By defining the estimation error as $(\epsilon_x \mid \epsilon_z)^T = (x \mid z)^T - (\hat{x} \mid \hat{z})^T$, the error between impulses of $\alpha(t)$ can be shown to obey the homogeneous differential equation

$$\begin{pmatrix} \dot{\epsilon}_x \\ \dot{\epsilon}_z \end{pmatrix} = \left[\begin{array}{c|c} A + K_1 C & FH \\ \hline K_2 C & D \end{array} \right] \begin{pmatrix} \epsilon_x \\ \epsilon_z \end{pmatrix} \ . \tag{153}$$

The gain matrices K_1, K_2 are designed so that $(\epsilon_x \mid \epsilon_z)^T \to 0$ "rapidly."

Using the plant/disturbance models given in Section III, Equation (153) is expanded to give

$$\begin{pmatrix} \dot{\epsilon}_{x1} \\ \dot{\epsilon}_{x2} \\ \dot{\epsilon}_z \end{pmatrix} = \left[\begin{array}{cc|c} k_{11} & 1 & 0 \\ 1+k_{12} & 0 & 1 \\ \hline k_{21} & 0 & 1 \end{array} \right] \begin{pmatrix} \epsilon_{x1} \\ \epsilon_{x2} \\ \epsilon_z \end{pmatrix} = A_2 \begin{pmatrix} \epsilon_x \\ \epsilon_z \end{pmatrix} \ . \tag{154}$$

The characteristic polynomial of A_2 is

$$\lambda^3 - (k_{11} + 1)\lambda^2 + (k_{11} - k_{12} - 1)\lambda + (1 + k_{12} - k_{21}) = 0 \ . \tag{155}$$

It is desired that the observer poles be located at

$$\lambda_1 = -4, \ \lambda_2 = -5, \ \lambda_3 = -7 \ , \tag{156}$$

which would result in a characteristic polynomial of

30

$$\lambda^3 + 16\lambda^2 + 83\lambda + 140 = 0 . \tag{157}$$

A comparison of Equations (157) and (155) permits the gains to be calculated as

$$k_{11} = -17, \ k_{12} = -101, \ k_{21} = -240 . \tag{158}$$

Using these gains, the observer is implemented as

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{z}} \end{pmatrix} = \begin{bmatrix} -17 & 1 & 0 \\ -100 & 0 & 1 \\ -240 & 0 & 1 \end{bmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{z} \end{pmatrix} - \begin{pmatrix} -17 \\ -101 \\ -240 \end{pmatrix} y + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u . \tag{159}$$

The observer used for the servo-command state estimates is the same one designed in Section V, Equation (113).

Figure 16 shows a block diagram of the plant/observer/controller composite system for the design of this Section. This system was simulated on a digital computer and several runs were made to check the performance of this system (see Table 3 for a simulation listing).

Figure 17 gives the tracking error for two cases: (1) $y_c = 0$, $w = 0$ and (2) $y_c = 0$, $w = e^t$. The curve for case (1) corresponds to the results shown in Figure 2 for the controller of Section V, and seems to yield about the same response as the case with $R = 0.001$ on Figure 2. Figures 18 and 19 show the output and the tracking error, respectively, for a case with $y_c = 1 + 0.1t$, $w = 0$. These results correspond to those of Figures 3 and 4. The settling times of the transients agree with the results in Figures 3 and 4 for the case with $R = 0.001$, but the magnitudes of the excursions are less for the controller of this Section. Figure 20 shows the output response for a case with $y_c = 1 + 0.1t$, $w = e^t$, corresponding to the results shown in Figure 6. The transients in this case are again smaller in magnitude than those in Figure 6. Figure 21 presents the tracking errors for four cases: (1) $y_c = 1 + 0.1t$; (2) $y_c = 2t$; (3) $y_c = -2-4t$; (4) $y_c = -10+10t$. These curves correspond to the data shown in Figures 7, 9, 10, and 11, respectively. The results are equivalent.

Several runs were made to check the sensitivity of the controller design of this section to variations in the gains. Figure 22 is a repeat of the case in Figure 16 with $y_c = 0$, $w = 0$, but with +5% and +10% variations in the gains shown on Figure 16. The results correspond to the results in Figure 12 for the design in Section V. As shown, the system in this section is not as sensitive, even to +10% gain variations, as was the system of Section V. In fact, as shown in Figures 23 and 24, the controller of this section can withstand gain variations from -25% to +50% without significant performance degradation.
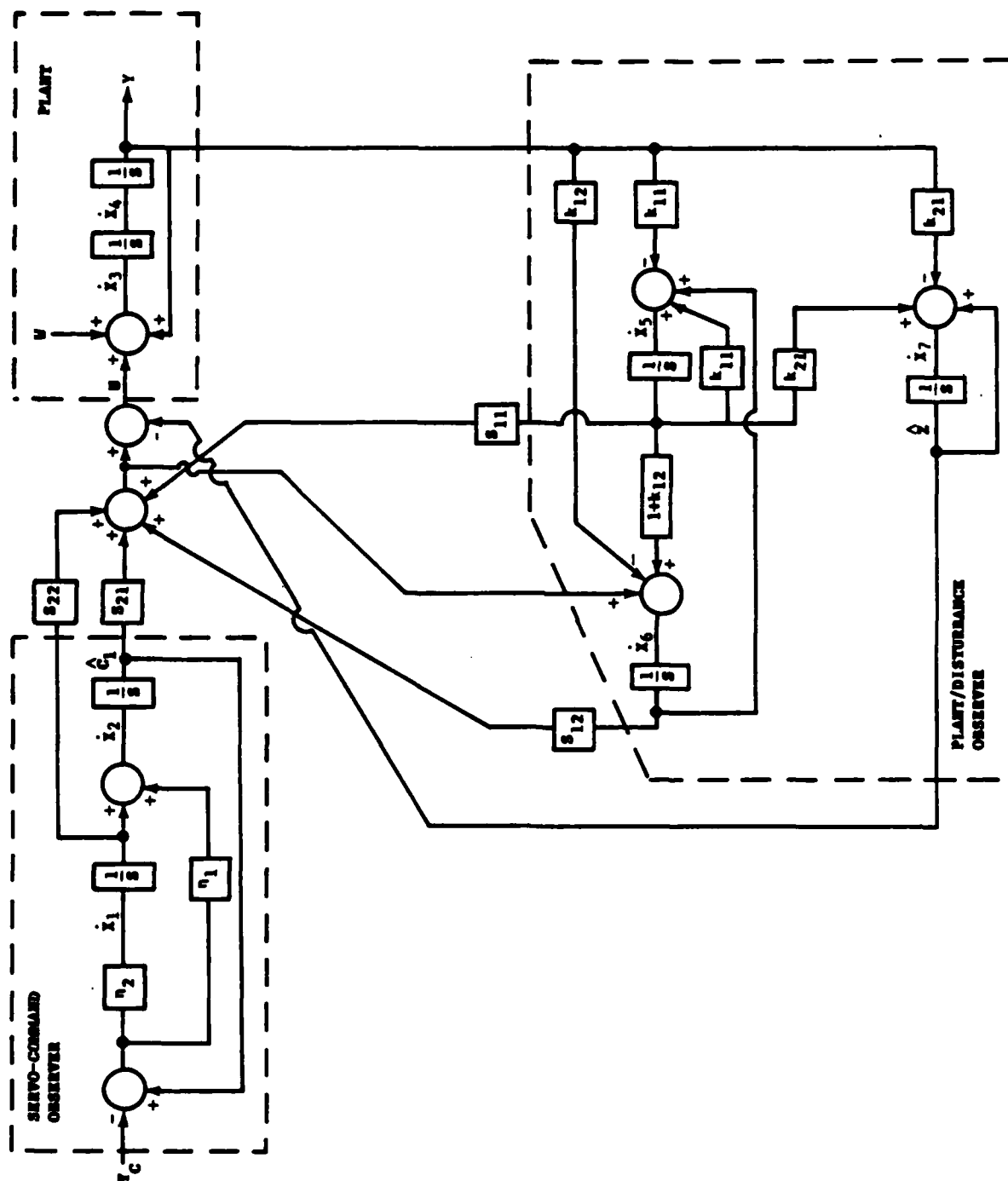
Figure 16. Composite system block diagram, technique 2.

32

TABLE 3. Simulation Listing for System of Figure 16

```
      COMMON X(7),DX(7),KUTTA,DT,NX
      DIMENSION XDAT(20)
      NVAR=16
      WRITE(20)NVAR
C ************************************************************
C     THIS PROGRAM SIMULATES A W = EXP(TIME)              *
C     WITH A STATE OBSERVER FOR A SERVO-PROBLEM           *
C     FOR JOHNSON'S METHOD 2                              *
C ************************************************************
      DO 100 I=1,7
      DX(I)=0.
      X(I)=0.
100   CONTINUE
      X(4) = 1.
      TIME=0.
      NX=7
      DT=0.01
      PRINT*,' ENTER WCO,WC1,YCO,YC1 '
      ACCEPT*,WCO,WC1,YCO,YC1
      PRINT*,' ENTER GAIN MULTIPLIER '
      ACCEPT*,GMPR
      XN1 = -16.*(1.+GMPR)


      XN2 = -64.*(1.+GMPR)
      S11 = -101.*(1.+GMPR)
      S12 = -20.*(1.+GMPR)
      XK11 = -17.*(1.+GMPR)
      XK12 = -101.*(1.+GMPR)
      XK21 = -240.*(1.+GMPR)
      S21 = 100.*(1.+GMPR)
      S22 = 20.*(1.+GMPR)
      UC = 0.
      US = 0.
      U = 0.
      W = 0.
      YC = 0.
      IPRT=0
1000  CONTINUE
      IF(TIME.GE.10.) GO TO 9999
      IPRT=IPRT+1
      DO 200 KUTTA=1,4
      W = WCO*EXP(WC1*TIME)
      YC = YCO + YC1*TIME
      SCERR = X(2) - YC
      SCTKE = -SCERR


      DX(1) = -64.*SCERR
      DX(2) = X(1) - 16.*SCERR
      US = S11*X(4) + S12*X(6) + S21*X(2) + S22*X(1)
      UC = -X(7)
      U = US + UC
      DX(3) = U + W + X(4)
      DX(4) = X(3)
      DX(5) = -XK11*X(4) + XK11*X(5) + X(6)
      DX(6) = (1.+XK12)*X(5) - XK12*X(4) + US
      DX(7) = -XK21*X(4) + XK21*X(5) + X(7)
      GO TO (30,60,30,40),KUTTA
30    CONTINUE
      TIME=TIME+.5*DT
40    CONTINUE
60    CALL RUNK
200   CONTINUE
      TRKERR = YC - X(4)
      EXER = W - X(7)
      XDAT(1)=X(1)
      XDAT(2)=X(2)
      XDAT(3)=X(3)
      XDAT(4)=X(4)
```

TABLE 3.   Simulation Listing for System of Figure 16 - Continued

```
      XDAT(5)=X(5)
      XDAT(6)=X(6)
      XDAT(7)=X(7)
      XDAT(8)=EXER
      XDAT(9)=YC
      XDAT(10)=TIME
      XDAT(11)=US
      XDAT(12)=UC
      XDAT(13)=X2HAT
      XDAT(14)=TRKERR
      XDAT(15)=SCTKE
      XDAT(16)=W
      IF(IPRT.NE.10) GO TO 500
      WRITE(20) (XDAT(I),I=1,NVAR)
      IPRT=0
      PRINT*,' TIME = ',TIME,' Y(T) = ',X(4),' YC = ',YC
C      PRINT*,' YC = ',YC,' YCHAT = ',X(2),' U = ',U
C      PRINT*,' W = ',W,' X2HAT = ',X2HAT,' X(2) = ',X(3)
500   GO TO 1000
9999  CONTINUE
      STOP
      END


      SUBROUTINE RUNK
      COMMON X(7),DX(7),KUTTA,DT,NX
      DIMENSION XA(7),DXA(7)
      GO TO (10,30,50,70),KUTTA
10    DO 20 I=1,NX
      XA(I)=X(I)
      DXA(I)=DT*DX(I)
20    X(I)=X(I)+.5*DXA(I)
      RETURN
30    TDT=2.*DT
      HDT=.5*DT
      DO 40 I=1,NX
      DXA(I)=DXA(I)+TDT*DX(I)
40    X(I)=XA(I)+HDT*DX(I)
      RETURN
50    DO 60 I=1,NX
      VDT=DT*DX(I)
      DXA(I)=DXA(I)+2.*VDT
60    X(I)=XA(I)+VDT
      RETURN
70    DO 80 I=1,NX
80    X(I)=XA(I)+(DXA(I)+DT*DX(I))/6.
```

34
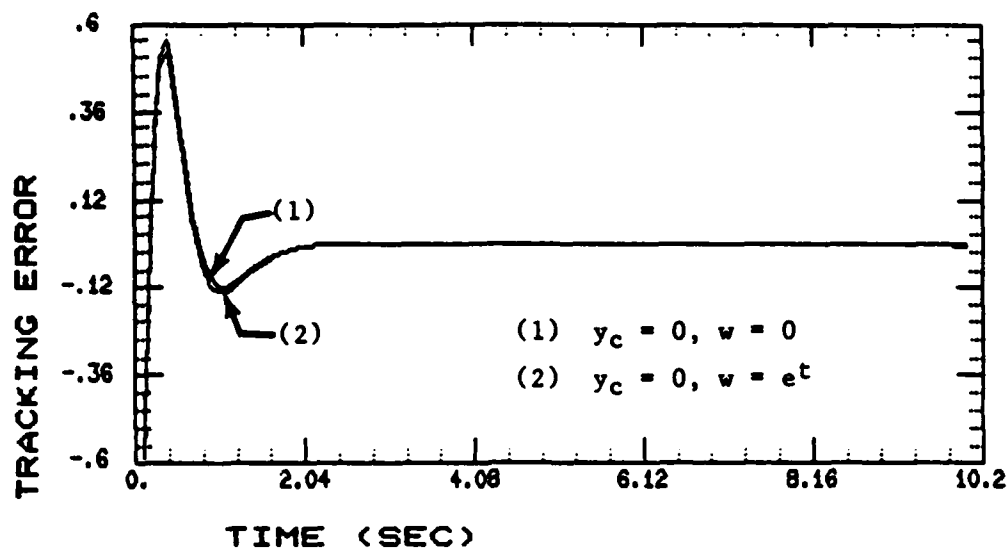
Figure 17. Servo-tracking error for cases with (1) $y_c = 0$, $w = 0$; (2) $y_c = 0$, $w = e^t$.
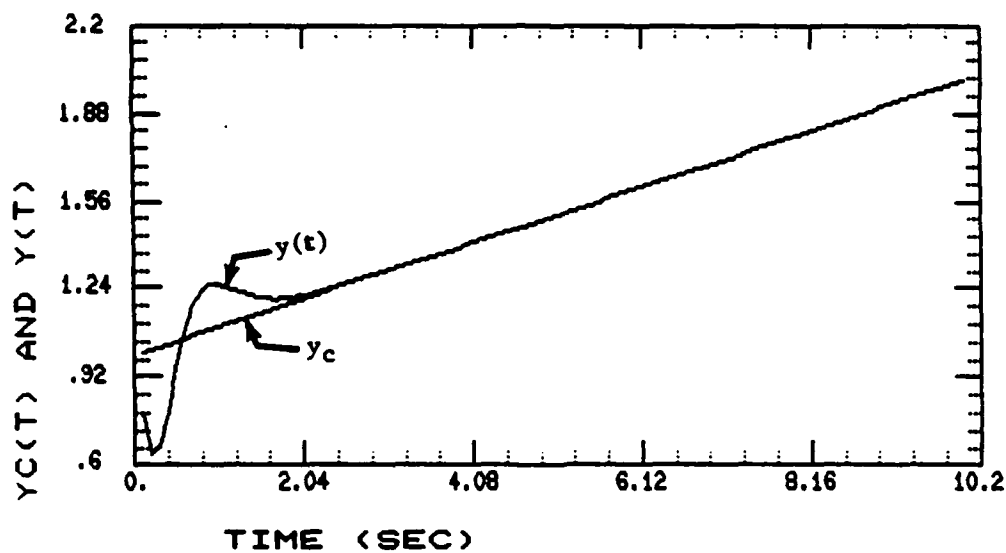
Figure 18. Servo-command input and plant output response for $y_c = 1 + 0.1t$, $w = 0$, $y(0) = 1$.

35
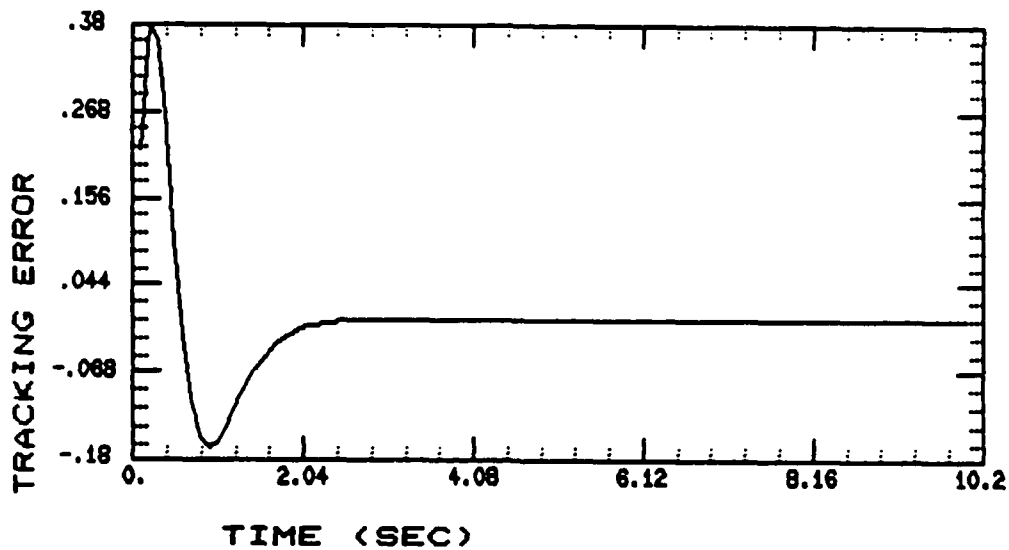
Figure 19.   Servo-tracking error corresponding to Figure 18.
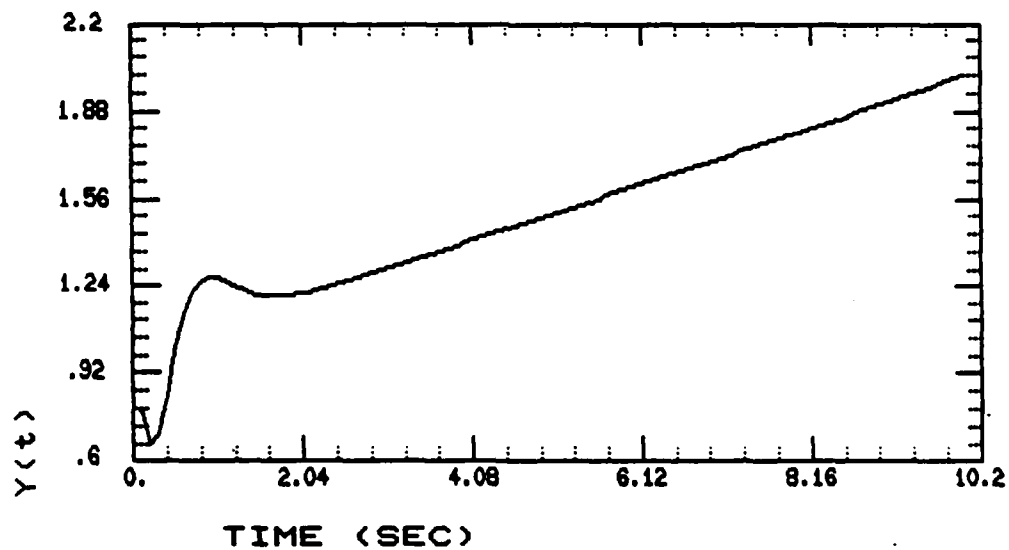
Figure 20.   Plant output response for a case with $y_c = 1 + 0.1t$, $w = e^t$, $y(0) = 1$.

36
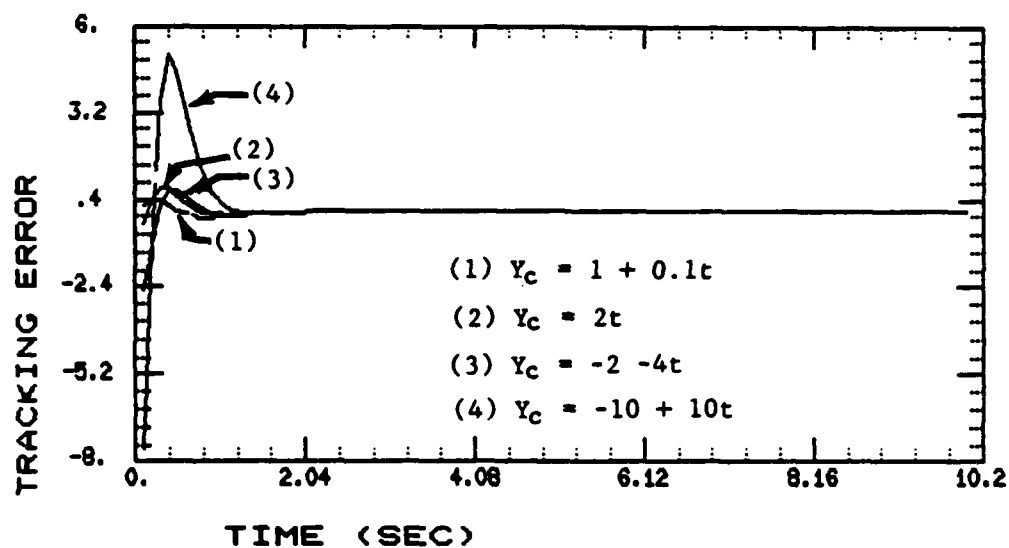
Figure 21. Servo-tracking error for four cases: (1) $y_c = 1 + 0.1t$; (2) $y_c = 2t$; (3) $y_c = -2 -4t$; (4) $y_c = -10 +10t$.
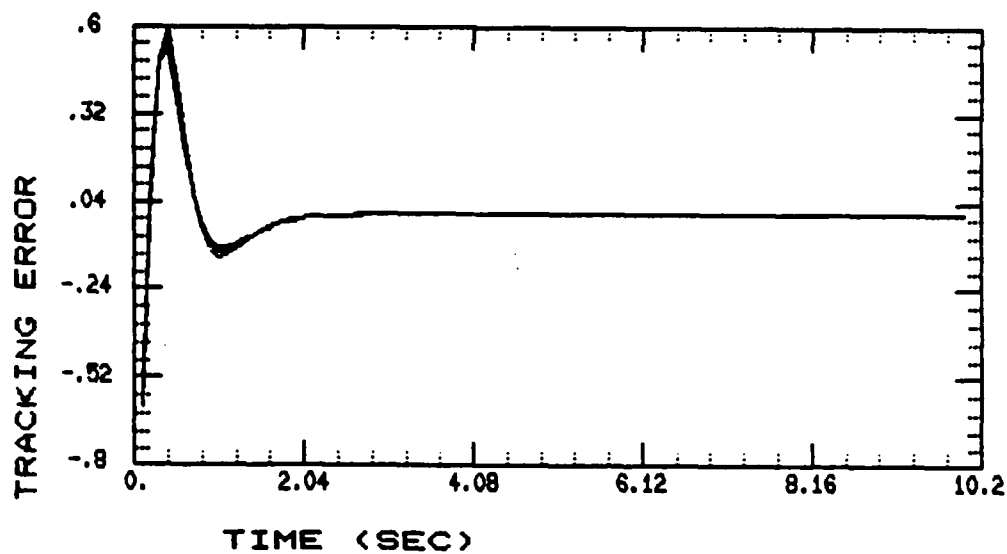
Figure 22. Servo-tracking error, for a case with $y_c = 0$, $w = 0$, for $\pm 5\%$ and $\pm 10\%$ system gain variations.

37
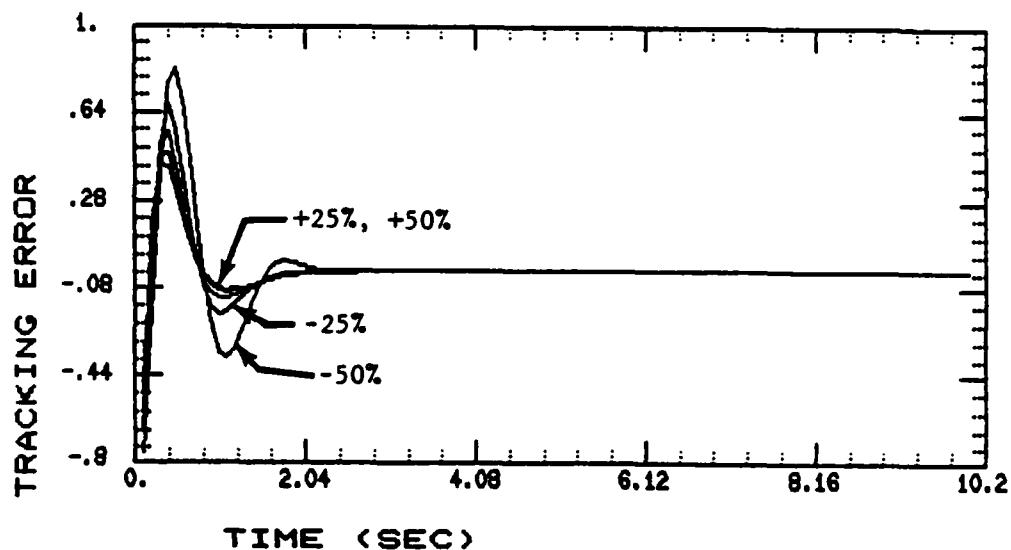
Figure 23.   Servo-tracking error, for a case with $y_c = 0$, $w = 0$,
for $\pm 25\%$ and $\pm 50\%$ system gain variations.
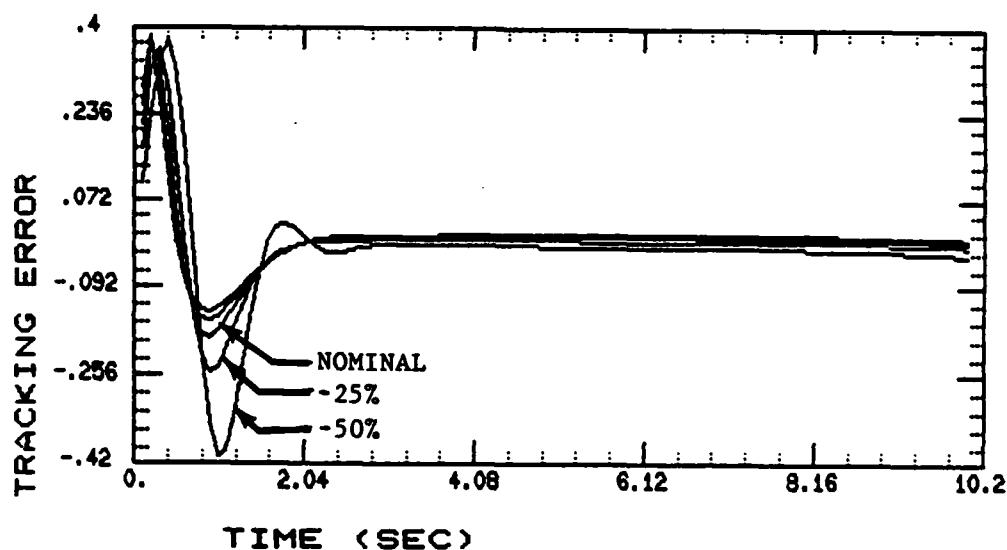
Figure 24.   Servo-tracking error, for a case with $y_c = 1 + 0.1t$,
$w = e^t$, with $\pm 25\%$ and $\pm 50\%$ system gain variations.

Two runs were made to investigate the robustness of this controller to differences between the actual and assumed external disturbance inputs to the plant. Figures 25 and 26 present the results, versus a reference curve for w $= e^t$, for cases with w $= e^{0.5t}$ and w $= e^{1.5t}$, respectively. As shown, this controller is also sensitive to variations from the assumed disturbance model.
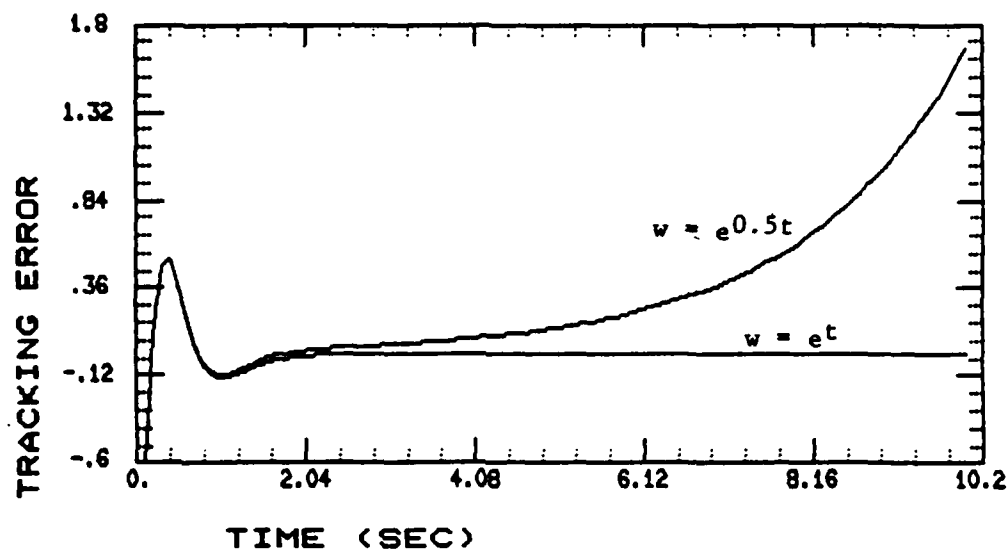
Figure 25. Servo-tracking with $y_c = 0$, $y(0) = 1$ for cases with: (1) w $= e^t$; (2) w $= e^{0.5t}$.
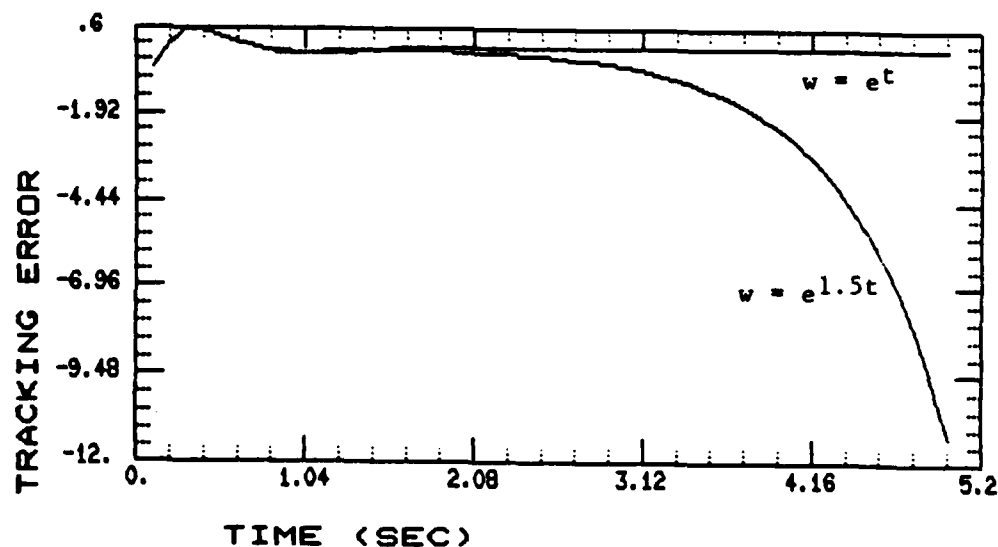
Figure 26. Servo-tracking error, with $y_c = 0$, $y(0) = 1$, for cases with: (1) w $= e^t$; (2) w $= e^{1.5t}$.

VII. DESIGN TECHNIQUE 3

In this section, the design technique proposed by Davison and Goldenberg in [8] and by Davison in [9] for design of a "robust feedback controller" is applied to the design of a controller for the plant of Section III. The robust feedback controller consists of [8] a "servo-compensator" and a "stabilizing compensator." This design technique assumes that the plant is linear and time-invariant and is described by the general form given in Equations (8) and (9). The tracking error is the difference between the output and the specified reference input and is given by Equation (10). The general form of the model for the external disturbance is given by Equations (11) and (12) and the general form for the servo-command input is given by Equations (13), (14) and (15).

The steps which comprise the design procedure for the robust feedback controller are as follows:

1. Check for satisfaction of the necessary and sufficient condition for the existence of a robust controller for the system. In order that a robust controller exist, the following conditions must all hold [8]:

      a. (A,B) must be stabilizable.

      b. (C_m,A) must be detectable, where $y_m = C_m x + D_m u + F_m \omega$ are the only outputs which are available for measurement.

      c. $m \geq r$ (160)

      d. The transmission zeros of (C,A,B,D) do not coincide with λ_i, $i = 1,2,\text{----},q$.

      e. y_m must contain the actual output y.

Conditions c and d are equivalent to satisfaction of [8]

$$\text{rank} \begin{bmatrix} A - \lambda_i I & B \\ C & D \end{bmatrix} = n + r \ , \ i = 1,2,\text{----},q. \qquad (161)$$

To perform this step, it is necessary to have the minimal polynomials of the matrices used to model the disturbances and servo-commands (A_1 and A_2 in Equations (11) and (14)). The λ_i are the zeros of the least common multiple of these two minimal polynomials.

The form of the controller, if one exists, is chosen as [8]

$$u = K_0 \hat{x} + K\xi \ , \qquad (162)$$

where ξ is the "servo-compensator" output and is an rq-vector and $\hat{x}$ is the "stabilizing compensator" output.

2. Design the servo-compensator, according to a recipe given in [8].

3. Since the plant in this example is unstable, a full-order stabilizing observer is designed and applied to it to provide a stable, satisfactory dynamic response.

4. The controller gains K_0 and K in Equation (162) are chosen so that the composite system has a satisfactory dynamic response and the complementary controller is implemented to provide the estimates of the plant states used in the control u.

To proceed with step 1, first determine the minimal polynomials of A_1 and A_2. The matrix A_1 is given in Equation (29) as $A_1 = 1$. The minimal polynomial of A_1, Λ_1, is found as

$$| \lambda I - A_1 | = \lambda - 1 , \tag{163}$$

$$\Lambda_1 = \lambda - 1 . \tag{164}$$

The matrix A_2 is given in Equation (35) as

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} . \tag{165}$$

and the minimal polynomial of A_2, Λ_2, is found as follows. The characteristic polynomial of A_2 is

$$| \lambda I - A_2 | = \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} = \lambda^2 . \tag{166}$$

Since A_2 is not a null matrix, but A_2^2 is a null matrix, the minimal polynomial of A_2 is given by

$$\Lambda_2 = \lambda^2 . \tag{167}$$

The least common multiple of Λ_1 and Λ_2 is

$$\Lambda = \lambda^2 (\lambda - 1) , \tag{168}$$

and the zeros are seen to be

$$\lambda_1 = 0, \ \lambda_2 = 0, \ \lambda_3 = 1 . \tag{169}$$

Proceeding with step 1.a., (A,B) must be checked for controllability. The pair (A,B) will be completely state controllable if, and only if, the composite matrix P_1, where

$$P_1 = [B | AB] , \tag{170}$$

is of rank n [11]. For this example then, with A and B as given in Equation (22),

$$P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{171}$$

$$\text{rank } [P_1] = 2 = n , \tag{172}$$

therefore, the pair (A,B) is completely state controllable.

From step 1.b., the pair (C,A) must be checked for observability. The pair (C,A) is completely observable if, and only if, the composite matrix P_2, where

$$P_2 = [C^T \mid A^T \, C^T] , \tag{173}$$

is of rank n [11]. For this example, with C given in Equation (23),

$$P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{174}$$

$$\text{rank}[P_2] = 2 = n , \tag{175}$$

therefore, the pair (C,A) is completely observable.

To check for satisfaction of the conditions in steps 1.c. and 1.d., Equation (161) will be used with the λ's of Equation (169). For $\lambda_1 = 0$, the result is

$$\text{rank } \begin{bmatrix} 0 & 1 & | & 0 \\ 1 & 0 & | & 1 \\ \hline 1 & 0 & | & 0 \end{bmatrix} = 3 = n + r = 3 . \tag{176}$$

For $\lambda_2 = 0$, the result is the same as in Equation (176). For $\lambda_3 = 1$, the result is

$$\text{rank } \begin{bmatrix} -1 & 1 & | & 0 \\ 1 & -1 & | & 1 \\ \hline 1 & 0 & | & 0 \end{bmatrix} = 3 = n + r = 3 . \tag{177}$$

Equation (161) is thus satisfied for each λ.

Since y, the actual output, is assumed to be measureable, step 1.e. is satisfied. Therefore, a robust controller exists for this system.

The next step in the design procedure is 2., the design of the servo-compensator. The general servo-compensator is described by [8]

$$\dot{\xi} = C^* \xi + B^* e, \tag{178}$$

where e is given by Equation (10), and [8]

$$B^* = \tau B \tag{179}$$

42

$$C^* = \tau \text{ block diag}(\underbrace{C, \ C, \ \text{----}, \ C}_{\text{r matrices}})\tau^{-1} \qquad (180)$$

with τ a non-singular real matrix; B a real matrix of rank r such that {block diag $(C, C, \text{----}, C)$, B} is controllable; and C is defined as a qxq companion matrix of the form [8]

$$C \triangleq \begin{bmatrix} 0 & 1 & 0 & \text{------} & 0 \\ 0 & 0 & 1 & \text{------} & 0 \\ | & | & | & | & | \\ L & | & | & | & | \\ | & | & | & | & | \\ | & | & | & | & | \\ -\delta_1 & -\delta_2 & -\delta_3 & \text{------} & -\delta_q \end{bmatrix} . \qquad (181)$$

The coefficients δ_1, δ_2, ----, δ_q in Equation (181) are the coefficients of the polynomial

$$\lambda^q + \delta_q \lambda^{q-1} + \text{----} + \delta_2 \lambda + \delta_1, \ \triangleq \ \prod_{i=1}^{q} (\lambda - \lambda_i), \qquad (182)$$

where the λ_i are the zeros of Λ. Expanding Equation (182) using the λ's of Equation (169) results in

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^2 (\lambda - 1) = \lambda^3 - \lambda^2 , \qquad (183)$$

and the coefficients δ_i are seen to be

$$\delta_1 = 0, \quad \delta_2 = 0, \quad \delta_3 = -1 . \qquad (184)$$

For this example then, the matrix C of Equation (181) is given by

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} . \qquad (185)$$

Following a suggestion in [8], let the matrix $\tau = I$ and let $B =$ block diag $(\gamma_1, \gamma_2, \text{----}, \gamma_r)$ where $\gamma_1 = \gamma_2 = \text{----} = \gamma_r = (0 \ 0 \ \text{---} \ 0 \ 1)^T$. In this case, since $r = 1$, one has

$$B^* = \tau B = IB = (0 \ 0 \ 1)^T , \qquad (186)$$

$$C^* = C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} . \qquad (187)$$

The servo-compensator is thus implemented according to Equation (178) as

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (y - y_{ref}) . \qquad (188)$$

43

The next step in the process is step 3, the design of the full-order stabilizing observer. The structure of this observer is given in [8] as

$$\dot{\rho} = (A - K_cC)\rho + K_c(y - Du) + Bu, \quad u = \overline{K}\rho + u^o \qquad (189)$$

and it is to be applied to the plant so as to provide a stable, satisfactory response, with K_c chosen so that $(A-K_cC)$ is stable. The parameter u^o is given by Equation (162). For this example, the matrix $(A-K_cC)$ can be expanded to give

$$(A-K_cC) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{pmatrix} k_{c1} \\ k_{c2} \end{pmatrix} (1, 0) = \begin{bmatrix} -k_{c1} & 1 \\ 1-k_{c2} & 0 \end{bmatrix} = \widetilde{A} \qquad (190)$$

and the characteristic polynomial of $\widetilde{A}$ is found as

$$| \lambda I - \widetilde{A} | = \begin{vmatrix} \lambda+k_{c1} & -1 \\ -1+k_{c2} & \lambda \end{vmatrix} = \lambda^2 + k_{c1}\lambda + (k_{c2} - 1) . \qquad (191)$$

Using pole placement techniques, place the roots of Equation (191) at

$$\lambda_1 = -4, \quad \lambda_2 = -5 , \qquad (192)$$

which would result in a characteristic polynomial of

$$\lambda^2 + 9\lambda + 20 = 0 . \qquad (193)$$

By comparing Equations (193) and (191), the gain components can be calculated to be

$$k_{c1} = 9, \quad k_{c2} = 21 . \qquad (194)$$

The observer given by Equation (189) can be implemented as

$$\begin{pmatrix} \dot{\rho}_1 \\ \dot{\rho}_2 \end{pmatrix} = \begin{bmatrix} -9 & 1 \\ -20 & 0 \end{bmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} + \begin{pmatrix} 9 \\ 21 \end{pmatrix} y + \begin{pmatrix} 0 \\ \overline{k}_1\rho_1 + \overline{k}_2\rho_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u^o. \qquad (195)$$

This observer was combined with the plant model (with $u^o=0$) and simulated on a digital computer. Appropriate values were chosen for the gain components $\overline{k}_1$, $\overline{k}_2$ to give adequate performance. These gains were chosen as

$$\overline{k}_1 = -40 , \quad \overline{k}_2 = -5 . \qquad (196)$$

The last step in the design is step 4 above, the evaluation of the gains K_0 and K for the controller u and the implementation of the complementary controller. The gains K_0 and K are to be chosen to provide a satisfactory dynamic response from the augmented system described by [8]

44

$$
\begin{pmatrix} \dot{x} \\ \dot{x}-\dot{\rho} \\ \dot{\xi} \end{pmatrix} = \begin{bmatrix} A+B\overline{K} & -B\overline{K} & 0 \\ 0 & A-K_cC & 0 \\ B^*(C+D\overline{K}) & -B^*D\overline{K} & C^* \end{bmatrix} \begin{pmatrix} x \\ x-\rho \\ \xi \end{pmatrix} + \begin{bmatrix} B \\ 0 \\ B^*D \end{bmatrix} u, \qquad (197)
$$

$$
u = (K_0, \ 0, \ K) \begin{pmatrix} x \\ x-\rho \\ \xi \end{pmatrix}, \qquad (198)
$$

$$
y = (C+D\overline{K}, \ -D\overline{K}, \ 0) \begin{pmatrix} x \\ x-\rho \\ \xi \end{pmatrix} + Du. \qquad (199)
$$

These gains are found, again, using pole placement techniques. By making the appropriate substitutions into Equation (193) an augmented system for this example is expressed as:

$$
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_1-\dot{\rho}_1 \\ \dot{x}_2-\dot{\rho}_2 \\ \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ k_{01}-39 & k_{02}-5 & 40 & 5 & k_1 & k_2 & k_3 \\ 0 & 0 & -9 & 1 & 0 & 0 & 0 \\ 0 & 0 & -20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_1-\rho_1 \\ x_2-\rho_2 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}. \quad (200)
$$

Let the matrix in Equation (200) be denoted by $\overline{A}$. The characteristic polynomial of $\overline{A}$ can be found to be

$$
\lambda^7 - (k_{02}-13)\lambda^6 - (k_{01}+8\,k_{02}-90)\lambda^5 - (8k_{01}+11\,k_{02}-k_3-347)\lambda^4
$$

$$
- (11k_{01}-20k_{02}+k_2+9k_3-329)\lambda^3 - (-20k_{01}+k_1+9k_2+20k_3+780)\lambda^2
$$

$$
- (9k_1+20k_2)\lambda - 20k_1 = 0 . \qquad (201)
$$

If it is desired that the roots of Equations (201) be placed at

$$
\lambda_1 = \lambda_2 = -4, \ \lambda_3 = \lambda_4 = -5, \ \lambda_5 = \lambda_6 = \lambda_7 = -7, \qquad (202)
$$

the gains can be solved for and the results are

$$K_0 = (k_{01}, k_{02}) = (-348., -26.) \tag{203}$$

$$K = (k_1, k_2, k_3) = (-6860., -6027., -2473.) \ . \tag{204}$$

The robust controller, as given by Equation (162), is

$$u^o = (-348., -26.) \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} + (-6860., -6027., -2473.) \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} . \tag{205}$$

The vector $\hat{x}$ is the output of the general complementary controller defined in [8] in the form

$$\dot{\hat{x}} = A\hat{x} + B\hat{u} \tag{206}$$

$$\dot{\hat{\rho}} = (A - K_c C)\hat{\rho} + K_c(\hat{y} - D\hat{u}) + B\hat{u} \tag{207}$$

$$\hat{y} = C\hat{x} + D\hat{u} \tag{208}$$

$$\hat{u} = \overline{K}\hat{\rho} + u^o \ . \tag{209}$$

A block diagram of the composite system, consisting of the plant (Equations (22) and (23)), the servo-compensator (Equation (188)), the full order stabilizing observer (Equations (195) and (196)), the complementary controller (Equations (206), (207), (208) and (209)) and the robust controller (Equation (205)), is shown in Figure 27.

A listing of the digital simulation for the system of Figure 27 is given in Table 4. A series of runs were made, using this digital simulation, to investigate the performance of the plant/controller system shown in Figure 27. Figure 28 gives the tracking error for cases with $y_{ref} = 0$, $w = 0$ and $y_{ref} = 0$, $w = e^t$. Figures 29 and 30 present the plant response and tracking error, respectively, for a case with $y_{ref} = 1 + 0.1t$, $w = 0$. Figures 31 and 32 show the plant response and tracking error, respectively, for $y_{ref} = 1 + 0.1t$, with and without the external disturbance. Figure 33 shows a comparison of the tracking errors for four cases with $w = e^t$: (1) $y_{ref} = 1 + 0.1t$; (2) $y_{ref} = 2t$; (3) $y_{ref} = -2-4t$; (4) $y_{ref} = -10+10t$ and Figure 34 shows the plant output response and servo-command input for the case with $y_{ref} = -10+10t$.

To investigate the sensitivity of the performance of the controller design of this section to variations in the system gains, a series of runs were made as discussed in the previous two sections. Figures 35 and 36 show the tracking error for a case with $y_{ref} = 0$, $w = 0$ for $\pm 5\%$, $\pm 10\%$, and $\pm 25\%$, $\pm 50\%$ gain variations, respectively. As shown in these figures, this controller is not very sensitive to the $\pm 10\%$ gain variations, however, the -50% variation just about doubles the system settling time. Figure 37 shows the results of $\pm 25\%$ and $\pm 50\%$ gain variations when $y_{ref} = 1 + 0.1t$, $w = e^t$.
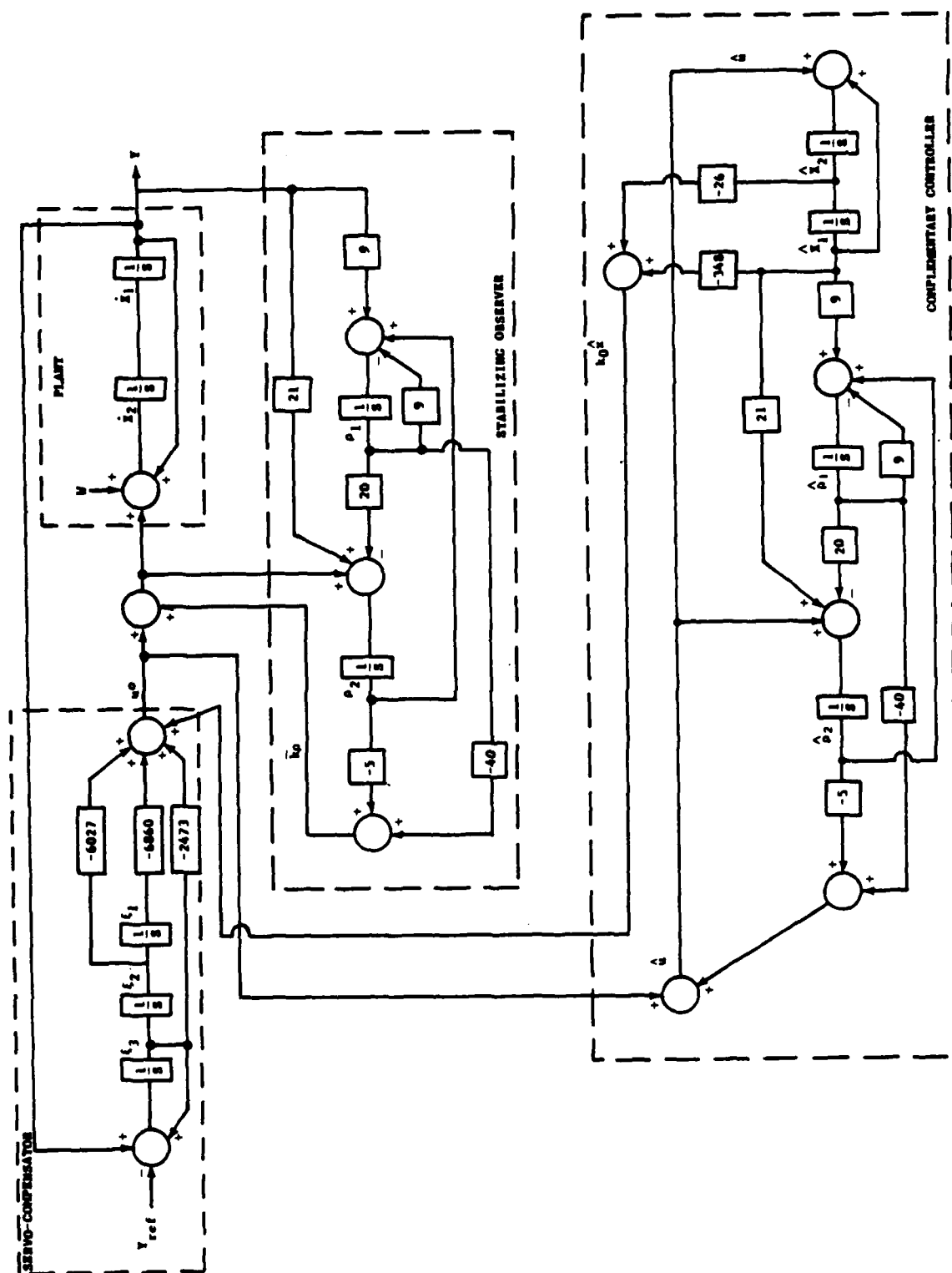
Figure 27. Composite system block diagram, technique 3.

47

TABLE 4. Simulation Listing for System of Figure 27

```
      COMMON X(11),DX(11),KUTTA,DT,NX
      DIMENSION XDAT(21)
      NVAR=21
      WRITE(20)NVAR
C **********************************************************
C       THIS PROGRAM SIMULATES A W = EXP(TIME)             *
C       WITH A STATE OBSERVER FOR A SERVO-PROBLEM          *
C       FOR DAVISON'S METHOD 1                             *
C **********************************************************
      DO 100 I=1,11
      DX(I)=0.
      X(I)=0.
100   CONTINUE
      X(5) = 1.
      TIME=0.
      NX=11
      DT=0.01
      PRINT*,' ENTER WC0,WC1,YC0,YC1 '
      ACCEPT*,WC0,WC1,YC0,YC1
      XK11 = -40.
      XK12 = -5.
      XKC1 = 9.


      XKC2 = 21.
      XKBR = 0.
      UCC = 0.
      UOPT = 0.
      UHAT = 0.
      U = 0.
      W = 0.
      YC = 0.
      IPRT=0
1000  CONTINUE
      IF(TIME.GE.10.) GO TO 9999
      IPRT=IPRT+1
      DO 200 KUTTA=1,4
      W = WC0*EXP(WC1*TIME)
      YREF = YC0 + YC1*TIME
      DX(1) = -YREF + X(1) + X(5)
      DX(2) = X(1)
      DX(3) = X(2)
      UOPT=-6027.*X(2)-6860.*X(3)-2473.*X(1)+UCC
      U = UOPT + XKBR
      DX(4) = U + W + X(5)
      DX(5) = X(4)


      DX(6) = XKC1*X(5) - XKC1*X(6) + X(7)
      DX(7) = (1.-XKC2)*X(6) + XKC2*X(5) + U
      XKBR = XK11*X(6) + XK12*X(7)
      UHAT = UOPT + XK11*X(10) + XK12*X(11)
      DX(8) = UHAT + X(9)
      DX(9) = X(8)
      UCC = -26.*X(8) - 349.*X(9)
      DX(10) = XKC1*X(9) + X(11) - XKC1*X(10)
      DX(11) = (1.-XKC2)*X(10) + XKC2*X(9) + UHAT
      GO TO (30,60,30,40),KUTTA
30    CONTINUE
      TIME=TIME+.5*DT
40    CONTINUE
60    CALL RUNK
200   CONTINUE
      TRKERR = YREF - X(5)
      XDAT(1)=X(1)
      XDAT(2)=X(2)  .
      XDAT(3)=X(3)
      XDAT(4)=X(4)
      XDAT(5)=X(5)
      XDAT(6)=X(6)
```

TABLE 4. Simulation Listing for System of Figure 27 - Continued

```
                    XDAT(7)=X(7)
                    XDAT(8)=X(8)
                    XDAT(9)=X(9)
                    XDAT(10)=X(10)
                    XDAT(11)=X(11)
                    XDAT(12)=TIME
                    XDAT(13)=DX(1)
                    XDAT(14)=UOPT
                    XDAT(15)=U
                    XDAT(16)=U
                    XDAT(17)=XKBR
                    XDAT(18)=YREF
                    XDAT(19)=UHAT
                    XDAT(20)=UCC
                    XDAT(21)=TRKERR
                    IF(IPRT.NE.10) GO TO 500
                    WRITE(20) (XDAT(I),I=1,NVAR)
                    IPRT=0
                    PRINT*,' TIME = ',TIME,' Y(T) = ',X(5),' YREF = ',YREF
          500       GO TO 1000
          9999      CONTINUE
                    STOP


                    END
                    SUBROUTINE RUNK
                    COMMON X(11),DX(11),KUTTA,DT,NX
                    DIMENSION XA(11),DXA(11)
                    GO TO (10,30,50,70),KUTTA
          10        DO 20 I=1,NX
                    XA(I)=X(I)
                    DXA(I)=DT*DX(I)
          20        X(I)=X(I)+.5*DXA(I)
                    RETURN
          30        TDT=2.*DT
                    HDT=.5*DT
                    DO 40 I=1,NX
                    DXA(I)=DXA(I)+TDT*DX(I)
          40        X(I)=XA(I)+HDT*DX(I)
                    RETURN
          50        DO 60 I=1,NX
                    VDT=DT*DX(I)
                    DXA(I)=DXA(I)+2.*VDT
          60        X(I)=XA(I)+VDT
                    RETURN
          70        DO 80 I=1,NX


          80        X(I)=XA(I)+(DXA(I)+DT*DX(I))/6.
                    RETURN
                    END
```

49
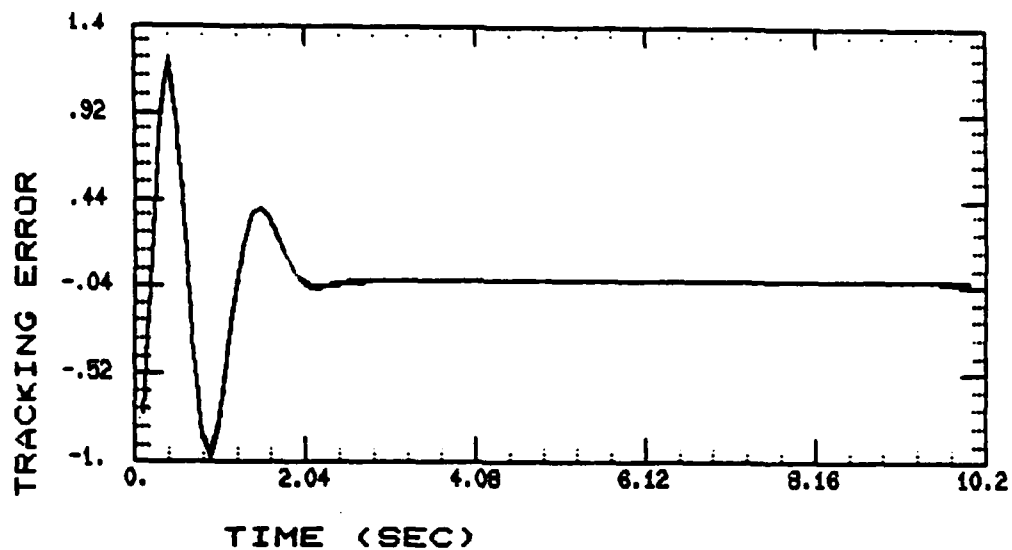
Figure 28. Servo-tracking error for cases with: (1) $y_{ref} = 0$, $w = 0$, $y(0) = 1$; (2) $y_{ref} = 0$, $w = e^t$, $y(0) = 1$.
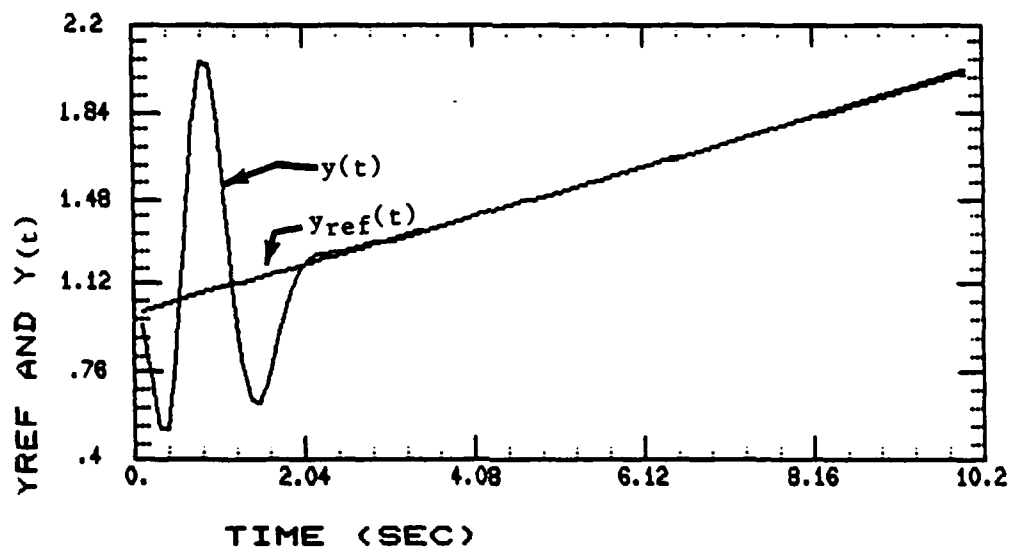
Figure 29. Servo-command input and plant output response for $y_{ref} = 1 + 0.1t$, $w = 0$, $y(0) = 1$.
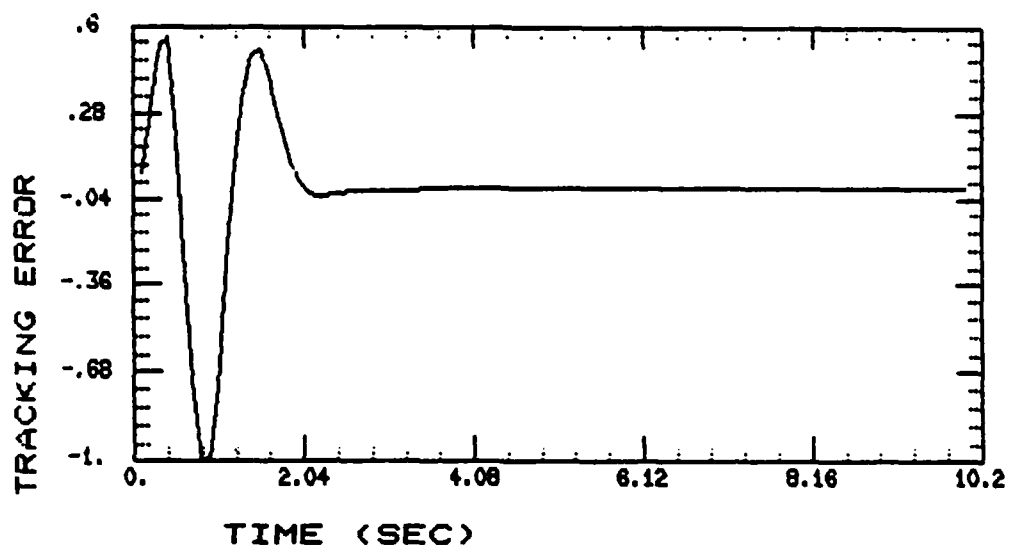
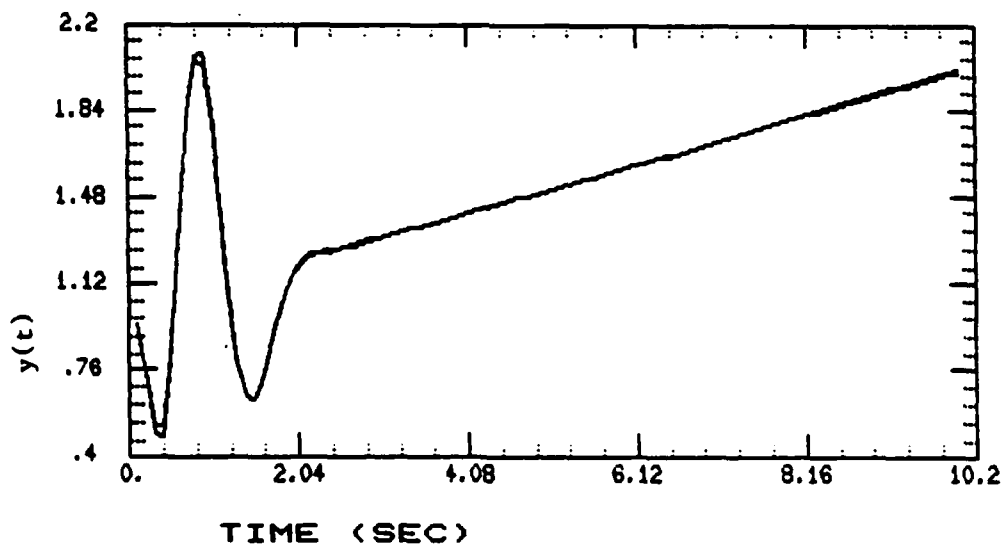Figure 30. Servo-tracking error corresponding to Figure 29.

Figure 31. Plant output response for a case with: (1) $y_{ref} = 1 + 0.1t$, $w = 0$, $y(0) = 1$; (2) $y_{ref} = 1 + 0.1t$, $w = e^t$. $y(0) = 1$.
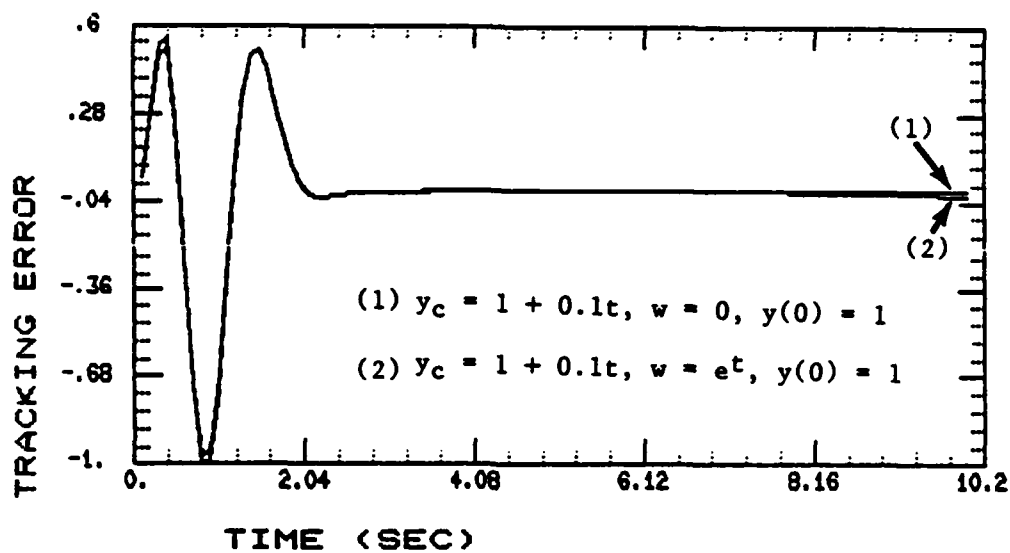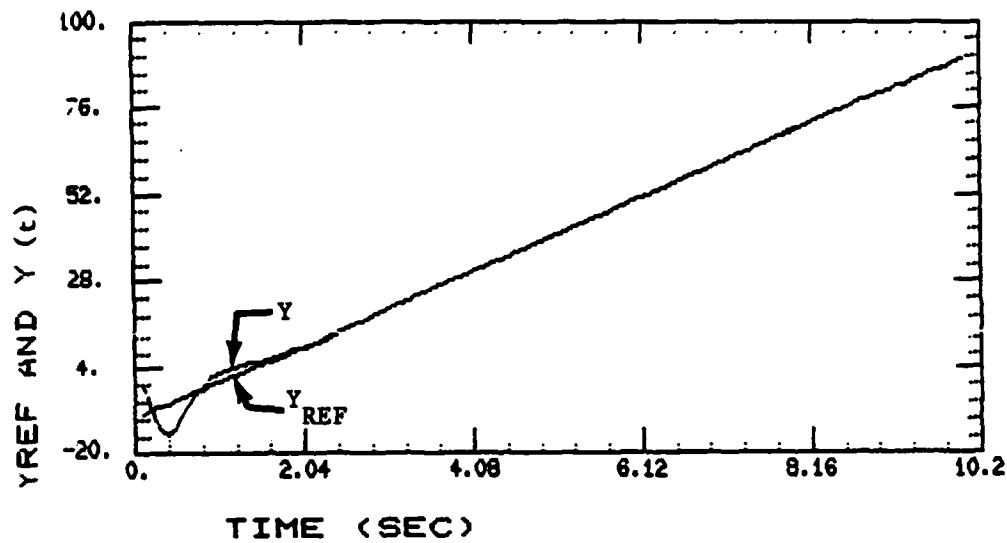
51

Figure 32. Servo-tracking error corresponding to Figure 31.
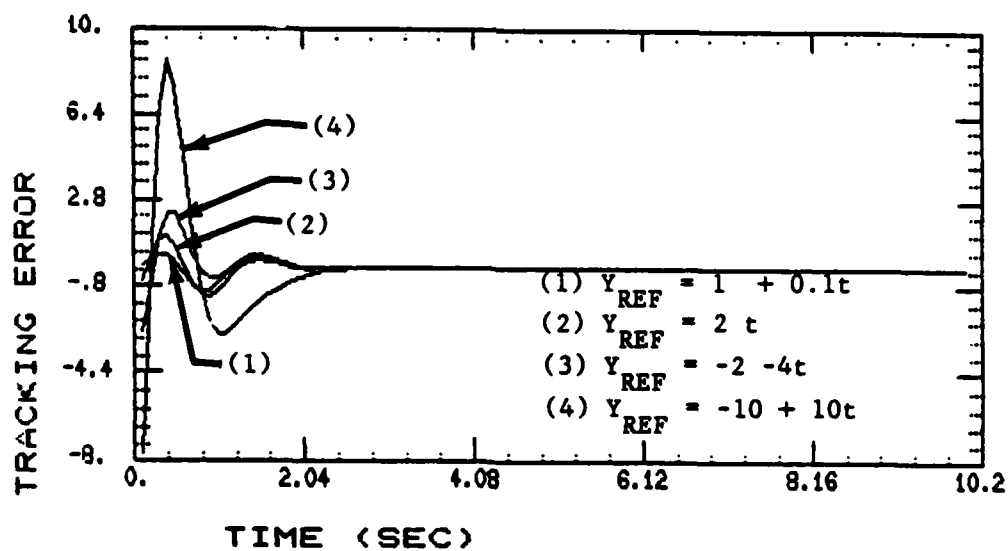
Figure 33. Servo-tracking error for four cases: (1) $y_{ref} = 1 + 0.1t$; (2) $y_{ref} = 2t$; (3) $y_{ref} = -2-4t$; (4) $y_{ref} = -10+10t$.

52

Figure 34. Servo-command input and plant output response for
$y_{ref} = -10 + 10t$, $w = e^t$, $y(0) = 1$.
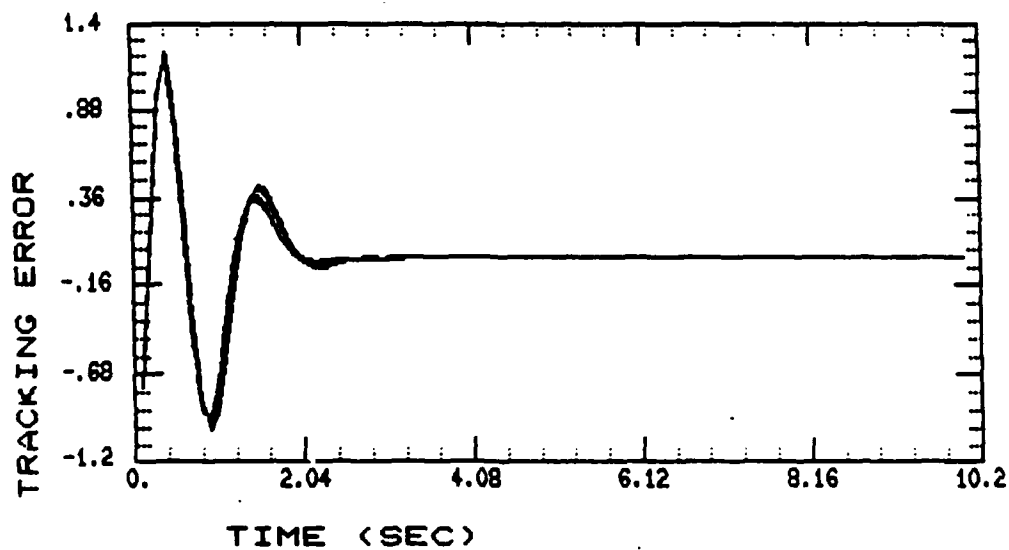
Figure 35. Servo-tracking error for a case with $y_{ref} = 0$, $w = 0$,
for $\pm 5\%$ and $\pm 10\%$ system gain variations.

53
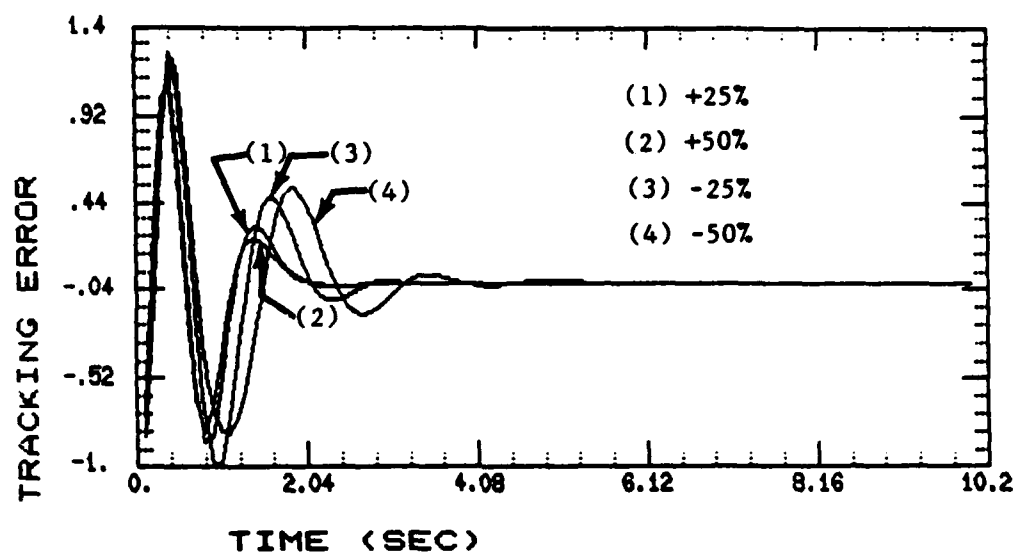
Figure 36.   Servo-tracking error for a case with $y_{ref} = 0$, $w = 0$,
for $\pm 25\%$ and $\pm 50\%$ system gain variations.
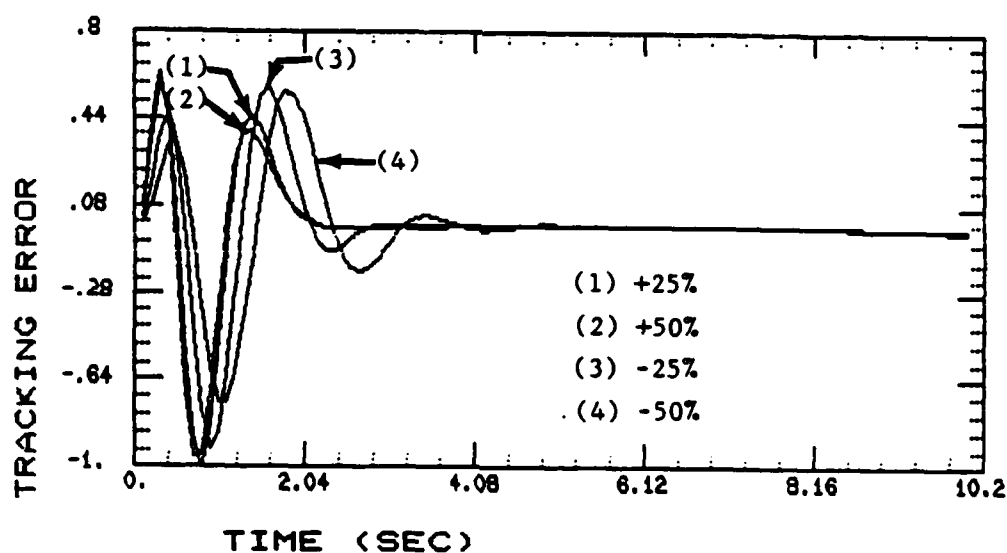
Figure 37.   Servo-tracking error for a case with $y_{ref} = 1 + 0.1t$,
$w = e^t$, $y(0) = 1$ for $\pm 25\%$ and $\pm 50\%$ system gain
variations.

Two additional runs were made to investigate the robustness of this controller to differences between the actual and assumed disturbance inputs to the plant. Figures 38 and 39, respectively, show the results for cases with $w = e^{0.5t}$ and $w = e^{1.5t}$, compared to a reference curve for a case with $w = e^t$. The results indicate that the controller of this section is less sensitive than the controllers of the previous two sections to these differences.

The plots presented in this section show that this controller design exhibits much larger initial transients than the controllers in Sections V and VI. Apparently this is due to the influence of the K term in the equation for u^o (Equations (162) and (205)). If this term is removed from u^o and the system is subjected to the initial condition $y(0) = 1$, the transient response is more like that exhibited by the other two controller designs. Figure 40 shows a comparison of the system response, for the design of this section, to the initial condition for a case with u^o of Equation (205) included and a case with $u^o = \hat{K_0}x$.
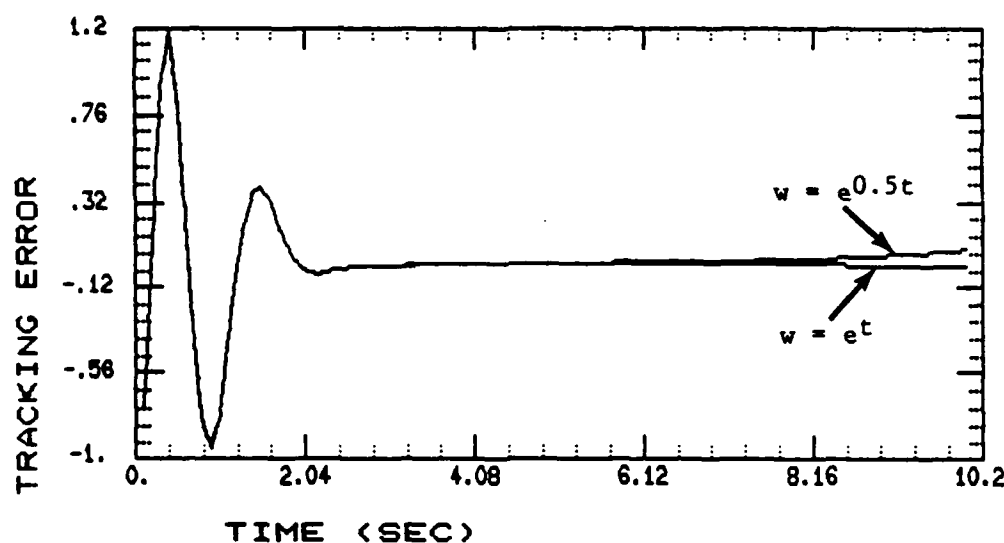
Figure 38. Servo-tracking error, $y_{ref} = 0$, $y(0) = 1$, for cases with:
(1) $w = e^t$; (2) $w = e^{0.5t}$.
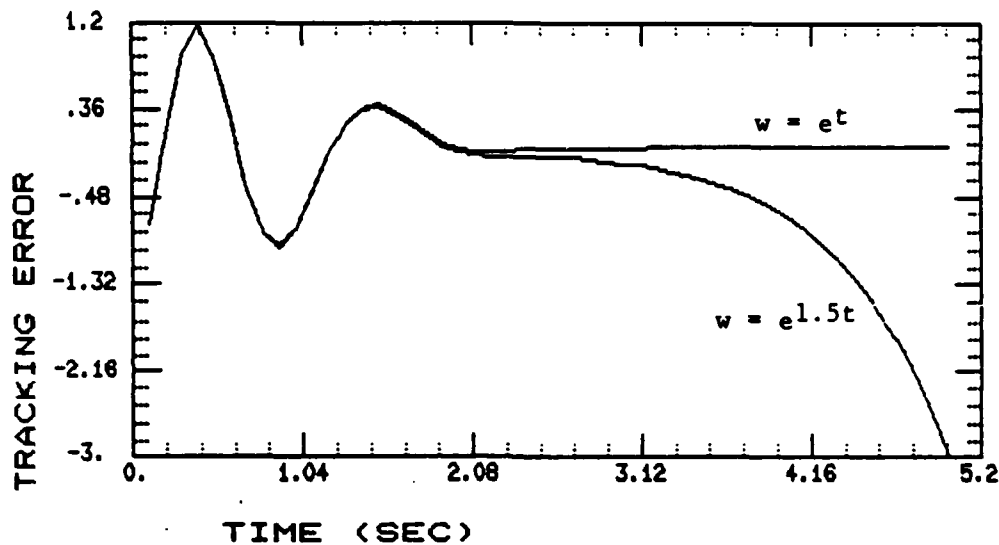
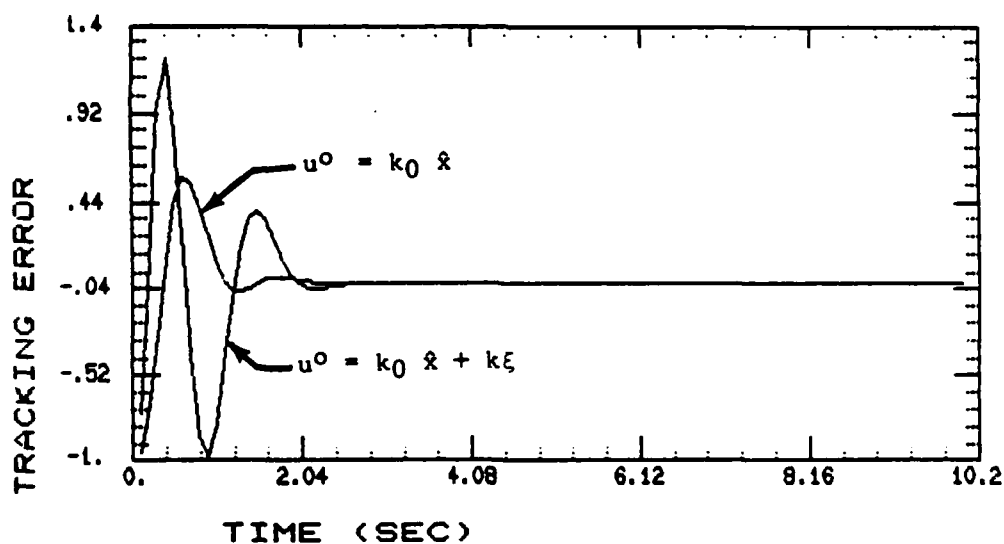Figure 39. Servo-tracking error, $y_{ref} = 0$, $y(0) = 1$, for cases with: (1) $w = e^t$; (2) $w = e^{1.5t}$.

Figure 40. Servo-tracking error for a case with $y_{ref} = 0$, $w = 0$, $y(0) = 1$ and with: (1) u^o as given by Equation (205); (2) u^o with only the $K_0\hat{x}$ term included.

56

VIII. SUMMARY AND CONCLUSIONS

The controller design procedures given in References [1,2,3,8 and 9] are all straightforward, in that, "design recipes" are given for calculating the required controller structures. However, the techniques all reflect the fact that as the order of the composite system increases, the matrix manipulations involved in the design processes become involved and tedious.

Johnson's linear algebraic design technique resulted in the simplest controller structure. Both of Johnson's design approaches, for the given example, resulted in an implementation with fewer additional integrators than Davison's technique. However, had the plant been stable, thus not requiring the inclusion of the stabilizing observer in the overall system, Johnson's optimal controller and Davison's controller would have been of about equal complexity. Johnson's optimal controller, with the reduced order observer and the required optimal gains, was computationally the most complex of the methods to implement.

As shown in the plots in Sections V, VI, and VII and the comparison plots in Appendix A, both of Johnson's designs gave comparable performance, including transient response. However, the design which used the linear algebraic approach gave the best overall performance. Davison's design exhibited the worst transient behavior, however, as indicated in Section VII, modification of the servo-compensator might improve this.

Johnson's optimal controller design proved to be the most sensitive to variations in the system gains, even to as little as 5 percent variation. The other two controllers were able to withstand up to 50 percent gain variations. All of the controllers were sensitive to differences between the actual and modeled external disturbance input.

When the external disturbance input to the plant was as modeled, all of the controllers were able to accommodate the effects of the external disturbance and achieve good servo-tracking out to t = 10 seconds. After about 10 seconds, none of the controller designs were able to completely negate the external disturbance effects. With no external disturbance included, all three controllers exhibited good servo-tracking when the simulation was run to t = 25 seconds, although in some cases, the controller in Section V began to drift.

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APPENDIX A

To help compare the performance of the controller designs of Sections V, VI and VII, the data shown on earlier plots in each of those sections were consolidated. Each plot in this Appendix contains three curves for each condition considered, one curve corresponding to each of the controller designs. Figures A-1 through A-7 compare the servo-tracking performance of the controllers for the various cases. Figures A-8 through A-13 compare the performance of the controllers when the system is subjected to gain variations. Figures A-14 and A-15 compare the performance of the controllers when the system is subjected to an external disturbance different from the assumed disturbance.

In addition, some results were generated to demonstrate the tracking performance of the controllers over a longer period of time than was shown in the previous plots. Figure A-16 is a comparison plot of the system performance, out to a time of 15 seconds, when the plant is subjected to an initial condition of $y(0) = 1$ and $w = e^t$. The performance of all the controllers, in the face of the external disturbance, is shown to be good out to about $t = 12$ seconds. Considering the magnitude of w at $t = 15$ seconds ($w = 3269020$) the performance indicated is very good but the trend in the magnitude of the servo-tracking error is obvious. Figures A-17 through A-22 show the performance of the controllers, with $w = 0$, over an interval of 25 seconds. The designs of Sections VI and VII performed well. The design of Section V, as shown by the data in Figures A-18, A-20 and A-22, allowed some error buildup.
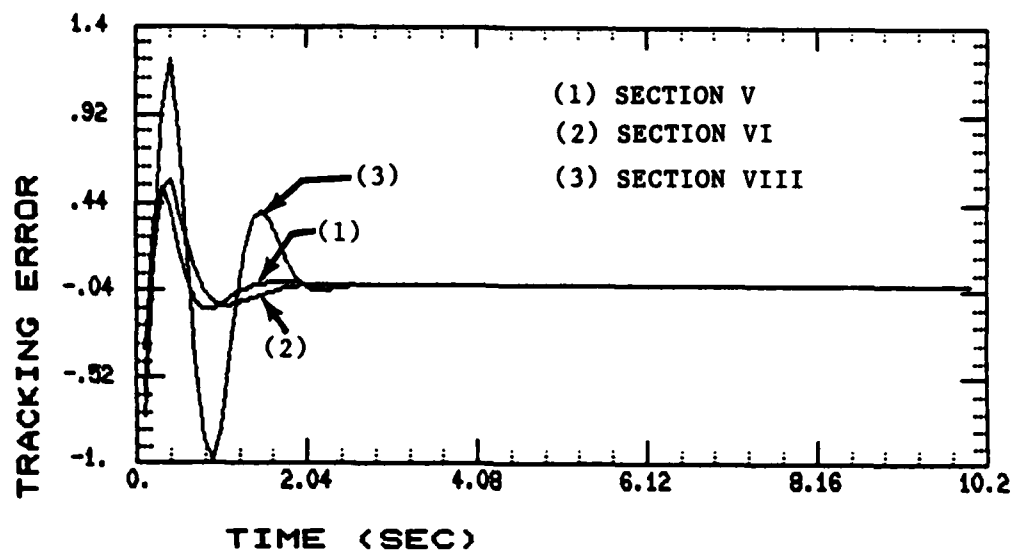
Figure A-1.   Servo-tracking error comparison with w = 0, y_c = 0, y(0) = 1.
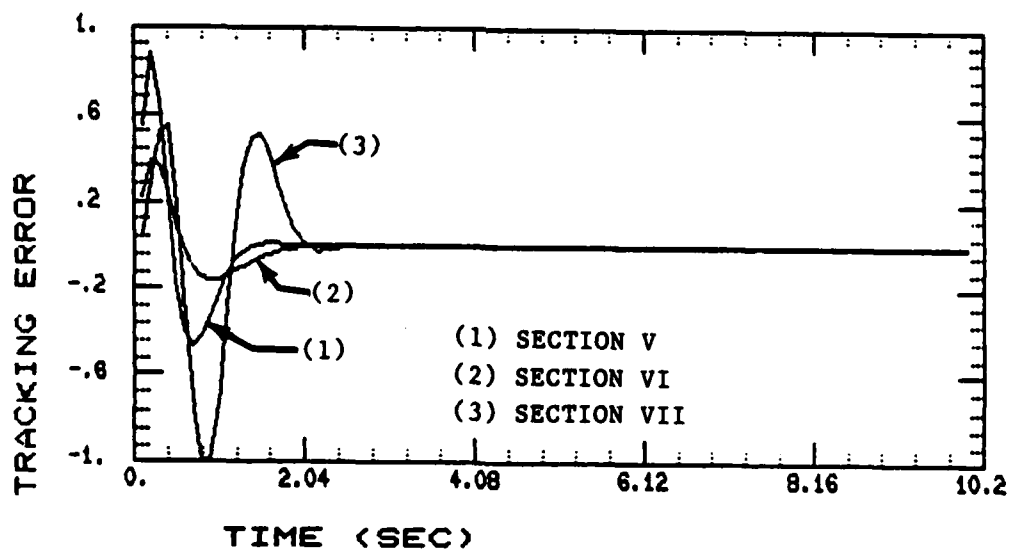
Figure A-2.   Servo-tracking error comparison with w = 0, y_c = 1 + 0.1t, y(0) = 1.
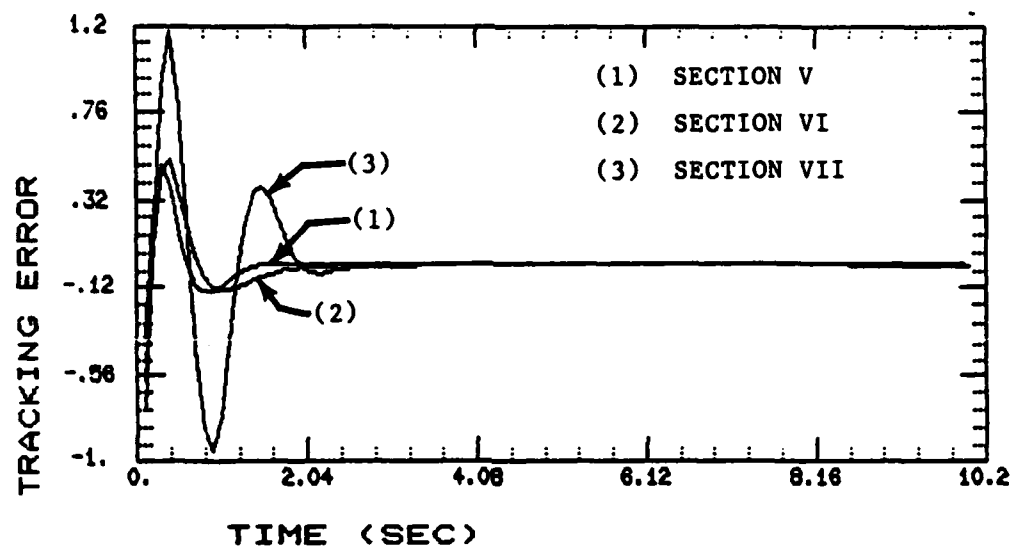
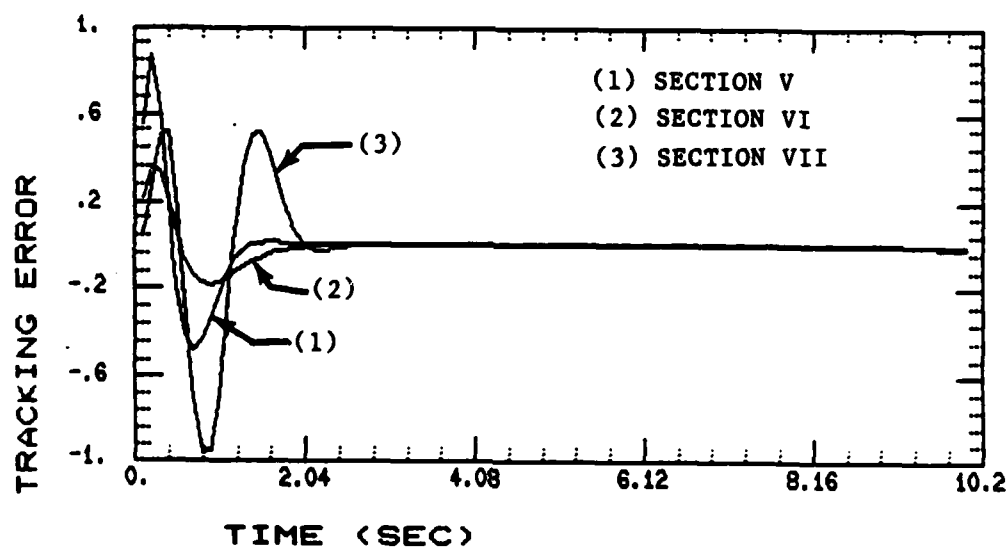Figure A-3. Servo-tracking error comparison with $w = e^t$, $y_c = 0$, $y(0) = 1$.

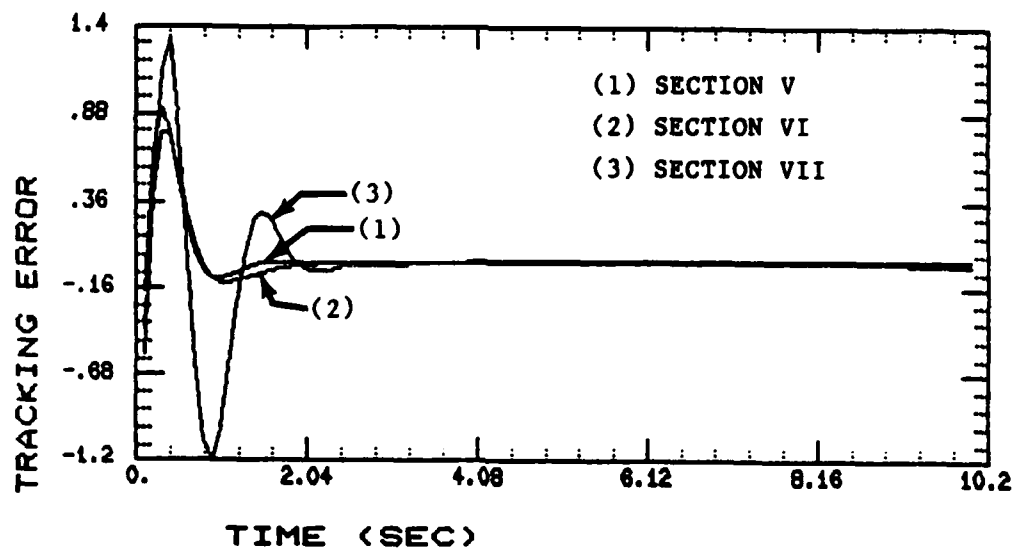Figure A-4. Servo-tracking error comparison with $w = e^t$, $y_c = 1 + 0.1t$, $y(0) = 1$.

A-3

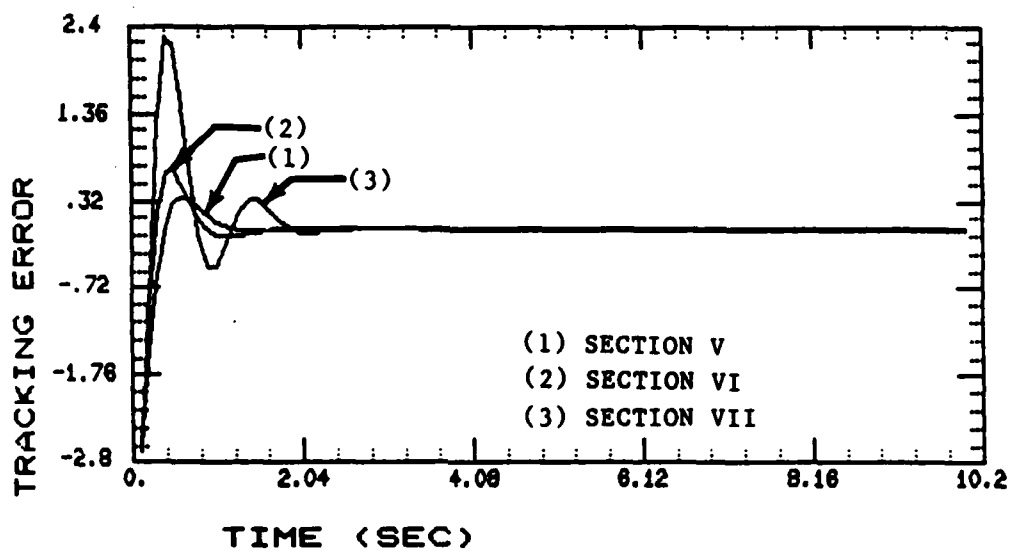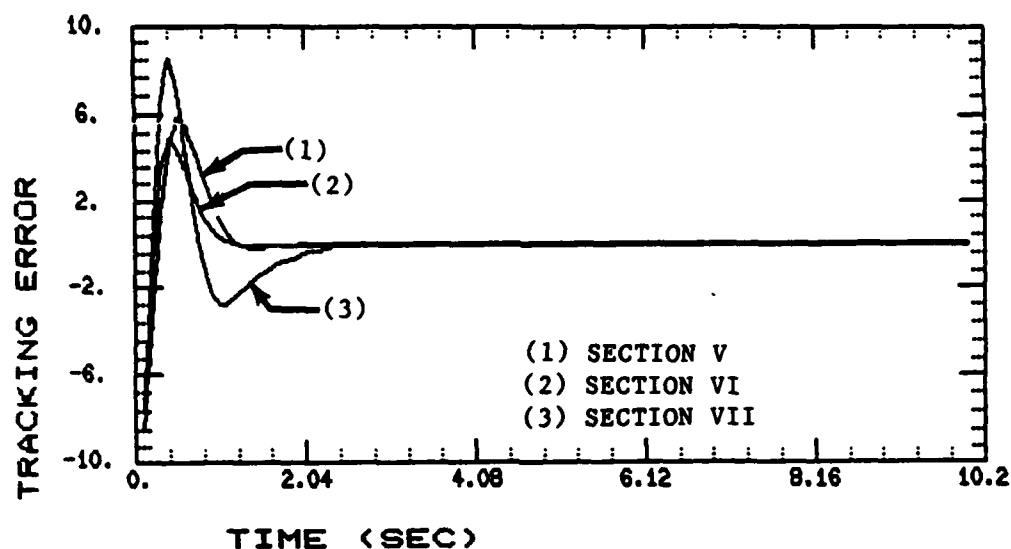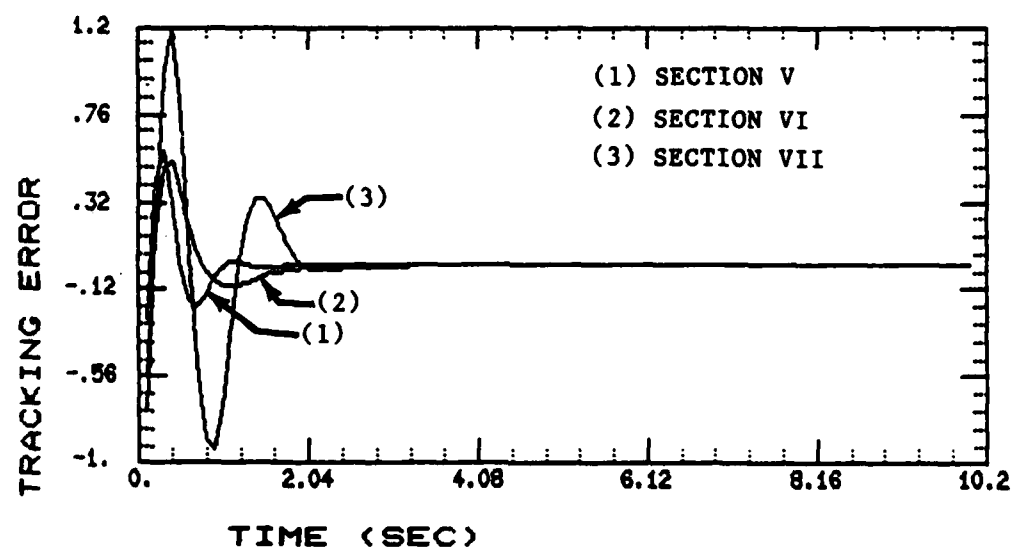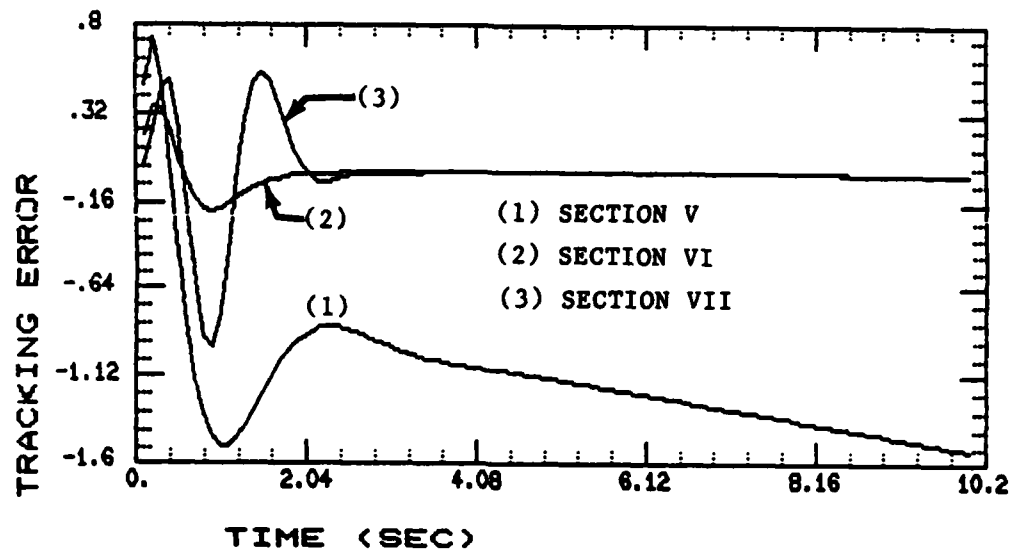Figure A-5. Servo-tracking error comparison with $w = e^t$, $y_c = 2t$, $y(0) = 1$.

Figure A-6. Servo-tracking error comparison with $w = e^t$, $y_c = -2 -4t$, $y(0) = 1$.

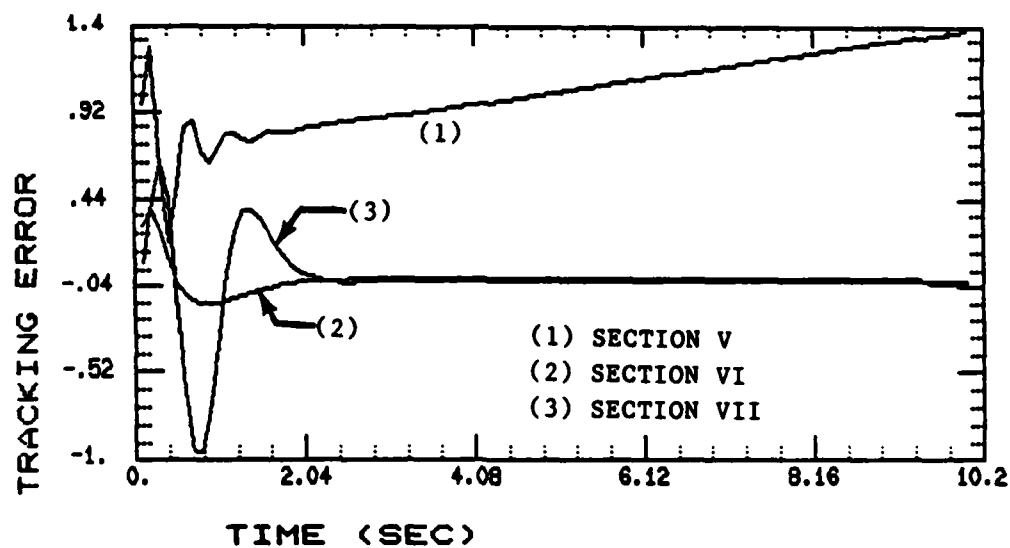Figure A-7. Servo-tracking error comparison with $w = e^t$, $y_c = -10 + 10t$, $y(0) = 1$.

Figure A-8. Servo-tracking error comparison, with +10% system gain variations, $w = 0$, $y_c = 0$, $y(0) = 1$.
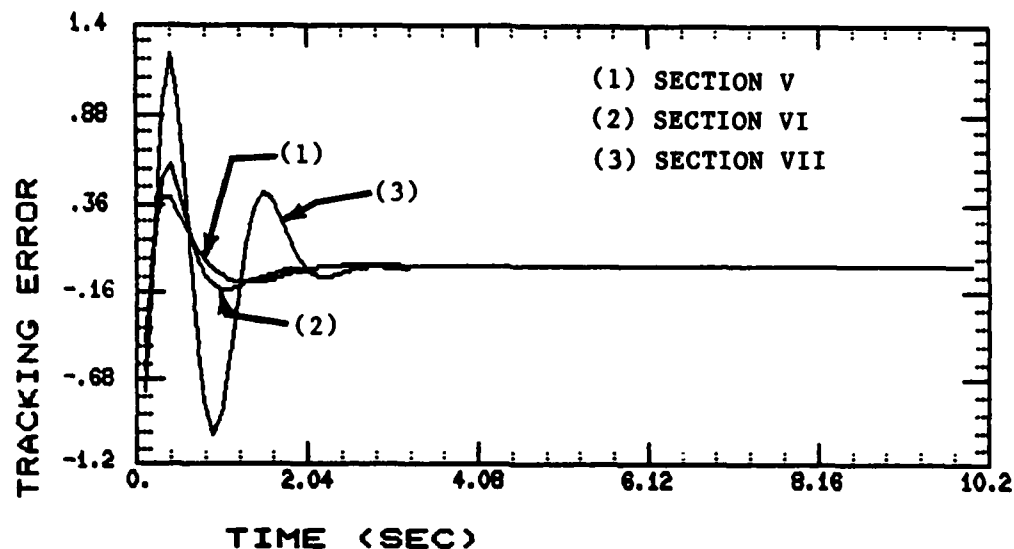
Figure A-9. Servo-tracking error comparison, with -10% system gain variations, w = 0, y_c = 0, y(0) = 1.
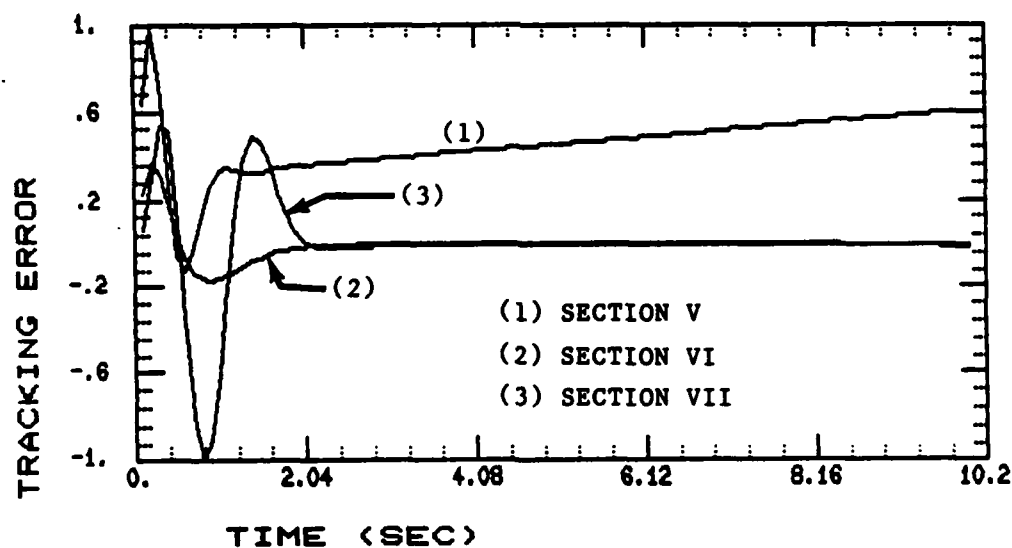
Figure A-10. Servo-tracking error comparison, with +10% system gain variations, w = e^t, y_c = 1 + 0.1t, y(0) = 1.
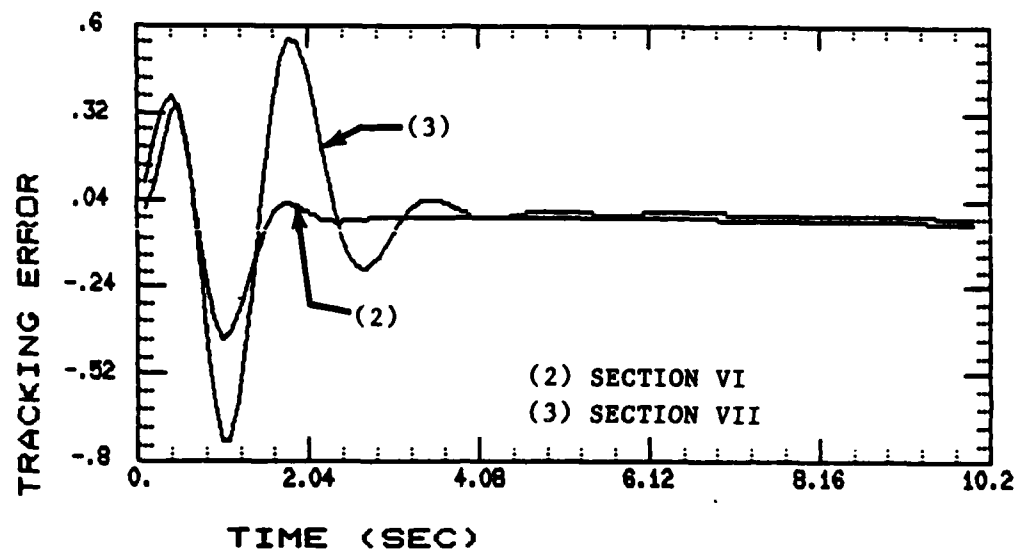
Figure A-11. Servo-tracking error comparison, with -10% system gain
variations, $w = e^t$, $y_c = 1 + 0.1t$, $y(0) = 1$.

Figure A-12. Servo-tracking error comparison, with +50% system gain
variations, $w = e^t$, $y_c = 1 + 0.1t$, $y(0) = 1$.

Figure A-13. Servo-tracking error comparison, with -50% system gain variations, $w = e^t$, $y_c = 1 + 0.1t$, $y(0) = 1$.
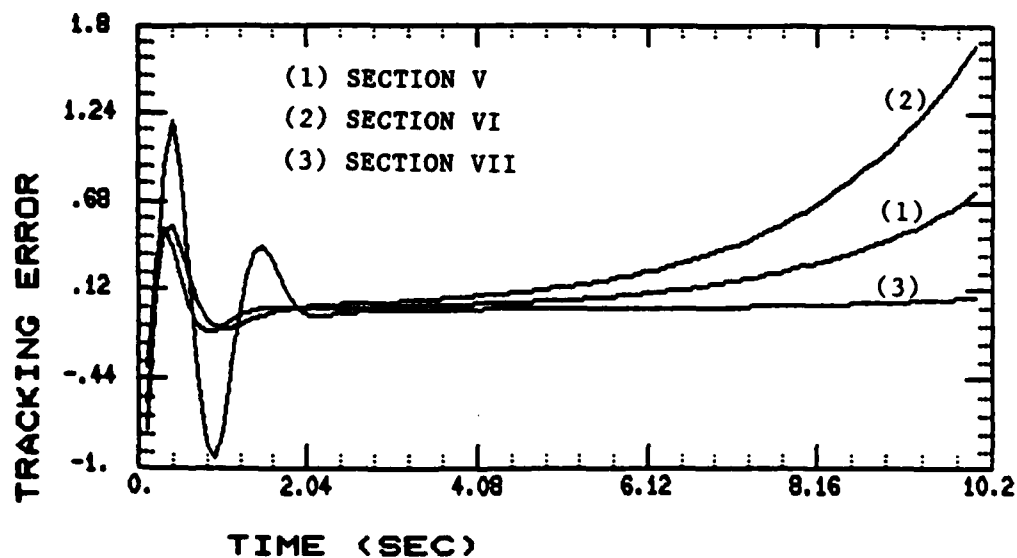

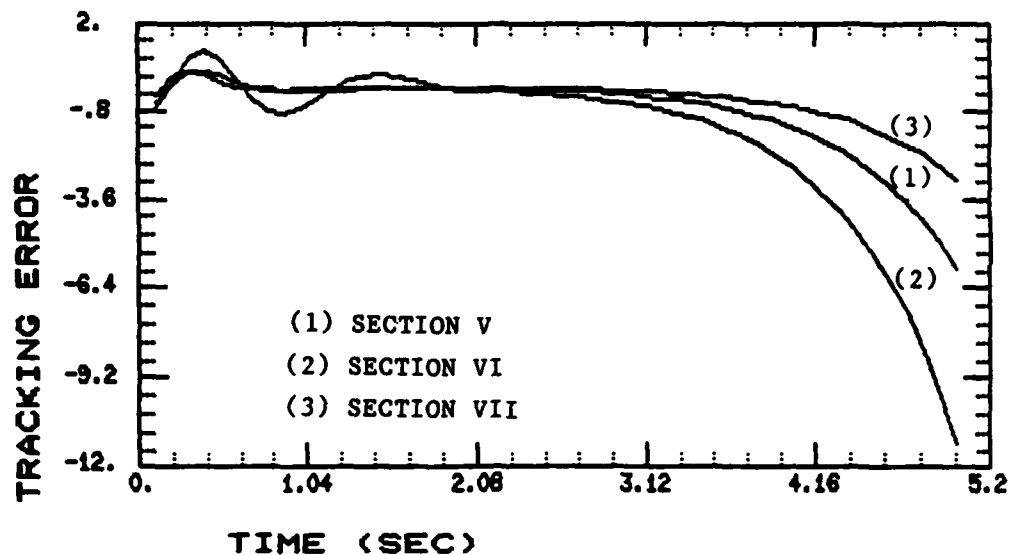Figure A-14. Servo-tracking error comparison with $y_c = 0$, $y(0) = 1$, $w = e^{0.5t}$.

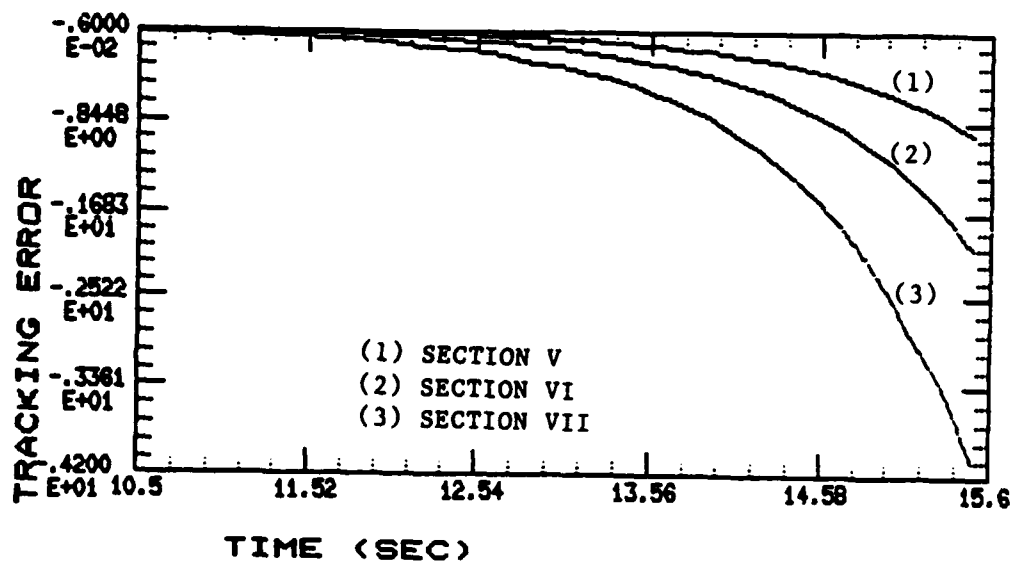Figure A-15. Servo-tracking error comparison with $y_c = 0$, $y(0) = 1$,
$w = e^{1.5t}$.

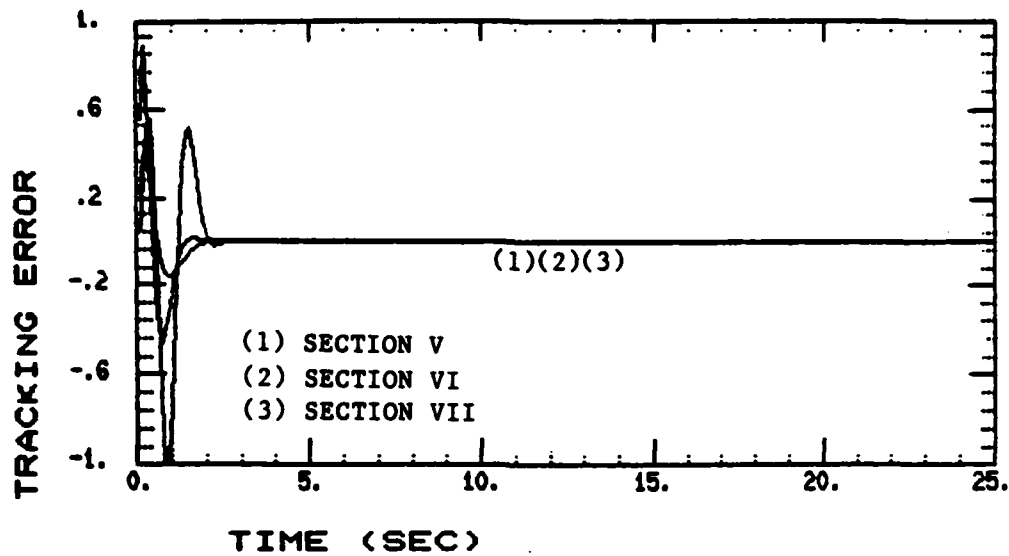Figure A-16. Servo-tracking error comparison with $y_c = 0$, $y(0) = 1$,
$w = e^t$.

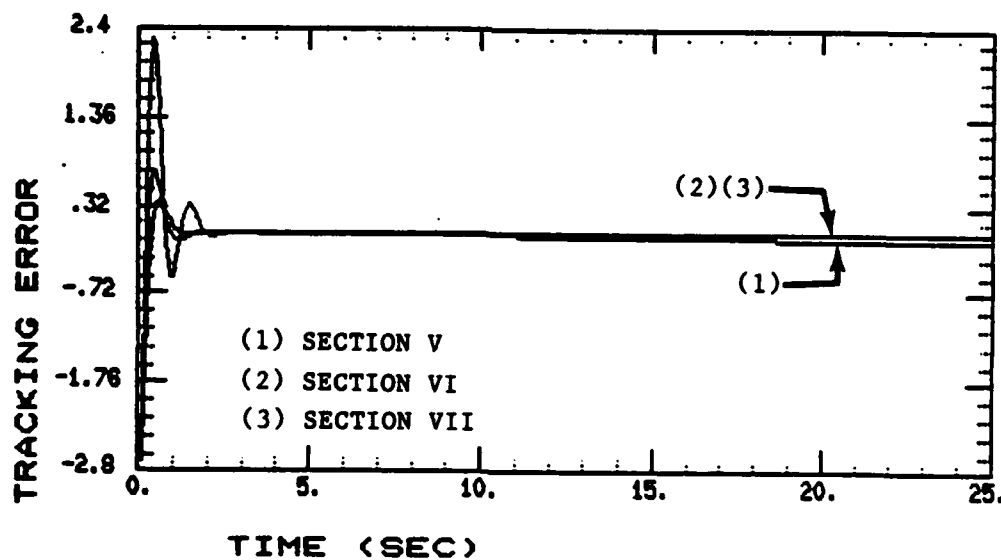Figure A-17. Servo-tracking error comparison with y(0) = 1, w = 0,
y_c = 1 + 0.1t.

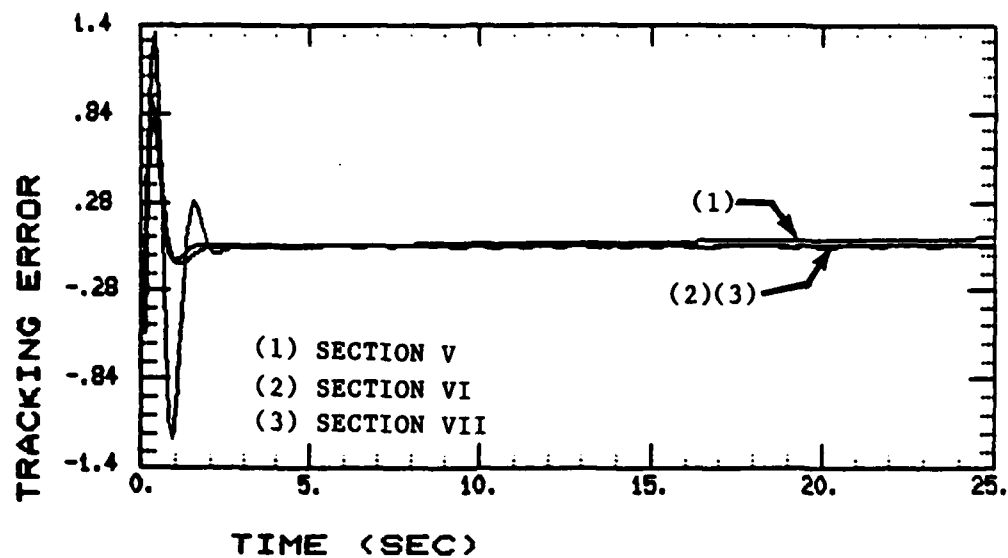Figure A-18. Servo-tracking error comparison with y(0) = 1, w = 0,
y_c = -2 -4t.

Figure A-19. Servo-tracking error comparison with $y(0) = 1$, $w = 0$, $y_c = 2t$.
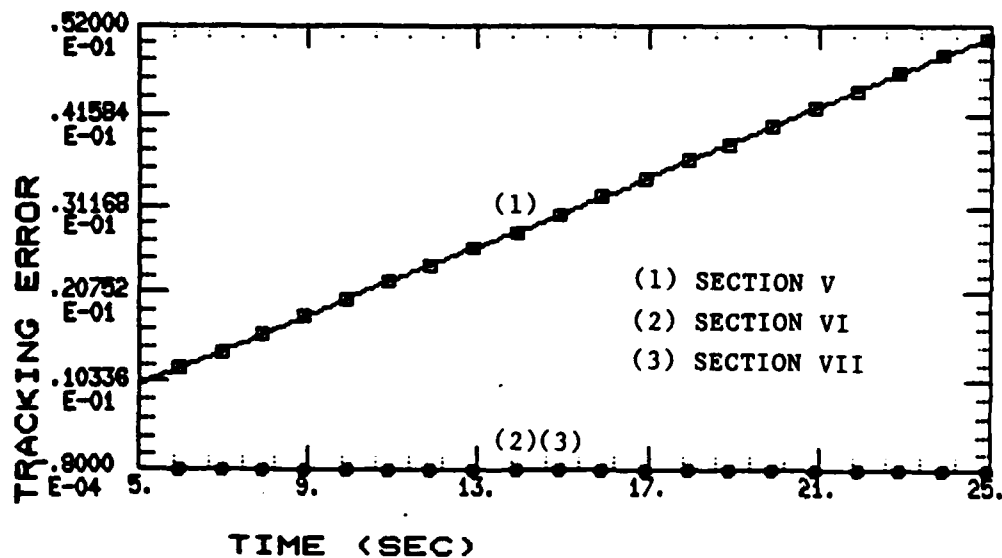

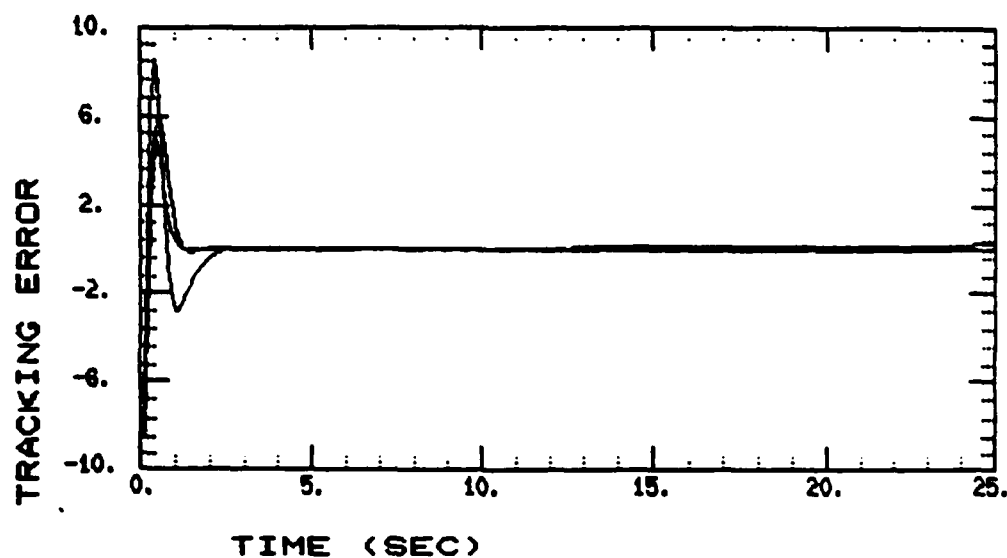Figure A-20. Expanded scale plot of a portion of Figure A-19.

Figure A-21. Servo-tracking error comparison with y(0) = 1, w = 0, y_c = -10 +10t.
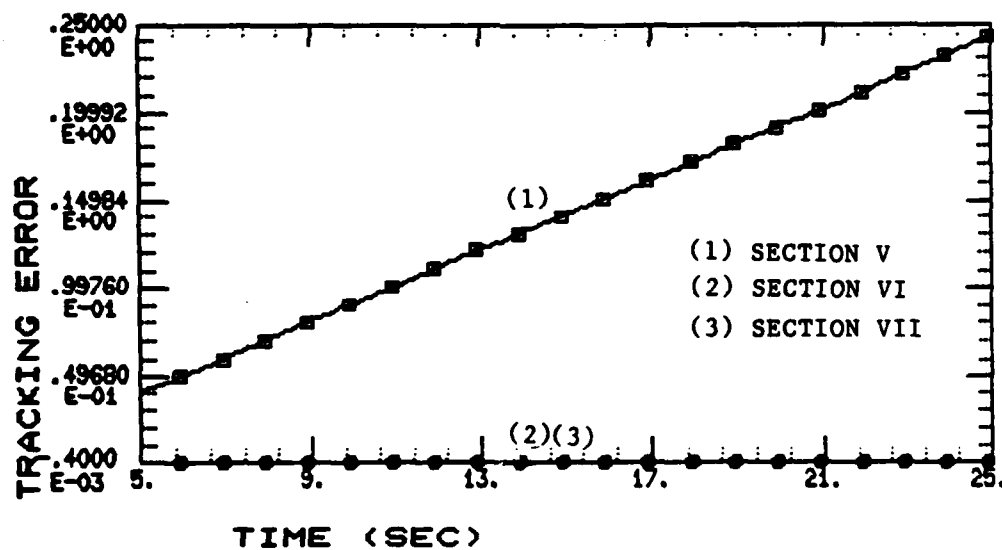

Figure A-22. Expanded scale plot of a portion of Figure A-21.

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